"Physics" department

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Newton’s Laws

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The proposed textbook is intended for independent work of undergraduate students in the field of "Technical Sciences and technology" in English. The textbook describes the basic provisions of the theory of physical laws, their technical application, also in the textbook describes the classical theory of mechanics, Newton's laws and their application.

This manual is intended for students of higher education institutions with the state language of study in physics.

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Introduction

Dynamic laws and laws of conservation of energy, momentum and angular momentum are the basic laws of mechanics. The study of them and is the content of this tutorial.

Newtonian mechanics is one of the most successful theories in the history of science; its success is based on Newton's three laws. The structure created by Newton system of mechanics served as the basis for accurate natural science of the XVIII and XIX centuries and now it fully meets the needs of quantitative calculations of mechanical movements. The first law establishes the presence of the property of inertia in material bodies and postulates the presence of such reference systems in which the movement of a free body occurs at a constant speed. Newton's second law on the basis of empirical facts postulates the relationship between the magnitude of the force, the acceleration of the body and its inertia. Newton's third law clarifies some properties of the concept of force introduced in the second law.

The presented tutorial outlines the basic provisions of the theory of physical laws, which can be tested by experience, as well as showing their technical application, describes the classical theory of mechanics, Newton's laws and their application. The tutorial consists of six chapters. The first Chapter is dealing with Newton's laws of motion. The second Chapter discusses the applicability of Newton's laws. In the third and fourth Chapter are the laws of conservation of energy and momentum, because the law of conservation of momentum is a consequence of Newton's laws for closed systems, and the law of conservation of energy is a consequence of Newton's laws for closed conservative systems. The fifth Chapter describes the dynamics of rotational motion. In the sixth Chapter the law of gravitation is considered.
The presentation of educational material is conducted without cumbersome mathematical calculations, due attention is paid to the physical essence of the phenomena and describing their concepts and laws.

In the presentation of the training material, a logical sequence is observed and the causal relationship between the phenomena is revealed.

The tutorial will contribute to the deepening of knowledge of students with an expanded program in the course of General physics, can serve as a basis for studying other technical disciplines.
1 Newton’s Laws of Motion

1.1 Force and Interactions

The principles of dynamics were clearly stated for the first time by Sir Isaac Newton (1642–1727); today we call them Newton’s laws of motion. The first law states that when the net force on a body is zero, its motion doesn’t change. The second law tells us that a body accelerates when the net force is not zero. The third law relates the forces that two interacting bodies exert on each other.

Newton did not derive the three laws of motion, but rather deduced them from a multitude of experiments performed by other scientists, especially Galileo Galilei (who died the year Newton was born). Newton’s laws are the foundation of classical mechanics (also called Newtonian mechanics); using them, we can understand most familiar kinds of motion. Newton’s laws need modification only for situations involving extremely high speeds (near the speed of light) or very small sizes (such as within the atom).

Newton’s laws are very simple to state, yet many students find these laws difficult to grasp and to work with. The reason is that before studying physics, you’ve spent years walking, throwing balls, pushing boxes, and doing dozens of things that involve motion. Along the way, you’ve developed a set of “common sense” ideas about motion and its causes. But many of these “common sense” ideas don’t stand up to logical analysis.

In everyday language, a force is a push or a pull. A better definition is that a force is an interaction between two bodies or between a body and its environment (Fig. 1.1). That’s why we always refer to the force that one body exerts on a second body. When you push on a car that is stuck in the snow, you exert a force on the car; a steel cable exerts a force on the beam it is hoisting at a construction site; and so on. As Fig. 1.1 shows, force is a vector quantity; you can push or pull a body in different directions.
Figure 1.1- Some properties of forces.

- A force is a push or a pull.
- A force is an interaction between two objects or between an object and its environment.
- A force is a vector quantity, with magnitude and direction.

When a force involves direct contact between two bodies, such as a push or pull that you exert on an object with your hand, we call it a contact force. Figures 1.2a, 1.2b, and 1.2c show three common types of contact forces. The normal force (Fig. 1.2a) is exerted on an object by any surface with which it is in contact. The adjective normal means that the force always acts perpendicular to the surface of contact, no matter what the angle of that surface. By contrast, the friction force (Fig. 1.2b) exerted on an object by a surface acts parallel to the surface, in the direction that opposes sliding. The pulling force exerted by a stretched rope or cord on an object to which it’s attached is called a tension force (Fig. 1.2c). When you tug on your dog’s leash, the force that pulls on her collar is a tension force.

In addition to contact forces, there are long-range forces that act even when the bodies are separated by empty space. The force between two magnets is an example of a long-range force, as is the force of gravity (Fig. 1.2d); the earth pulls a dropped object toward it even though there is no direct contact between the object and the earth. The gravitational force that the earth exerts on your body is called your weight.

To describe a force vector $\vec{F}$, we need to describe the direction in which it acts as well as its magnitude, the quantity that describes “how much” or “how hard” the force pushes or pulls. The SI unit of the magnitude of force is the newton, abbreviated N. Table 1 lists some typical force magnitudes.
(a) **Normal force** $\vec{N}$: When an object rests or pushes on a surface, the surface exerts a push on it that is directed perpendicular to the surface.

(b) **Friction force** $\vec{F}$: In addition to the normal force, a surface may exert a friction force on an object, directed parallel to the surface.

(c) **Tension force** $\vec{T}$: A pulling force exerted on an object by a rope, cord, etc.

(d) **Weight** $\vec{P}$: The pull of gravity on an object is a long-range force (a force that acts over a distance).

### Table 1 - Typical Force Magnitudes

<table>
<thead>
<tr>
<th>Description</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun’s gravitational force on the earth</td>
<td>$3.5 \times 10^{22}$ N</td>
</tr>
<tr>
<td>Weight of a large blue whale</td>
<td>$1.9 \times 10^6$ N</td>
</tr>
<tr>
<td>Maximum pulling force of a locomotive</td>
<td>$8.9 \times 10^5$ N</td>
</tr>
<tr>
<td>Weight of a 250-lb linebacker</td>
<td>$1.1 \times 10^3$ N</td>
</tr>
<tr>
<td>Weight of a medium apple</td>
<td>1 N</td>
</tr>
<tr>
<td>Weight of the smallest insect eggs</td>
<td>$2 \times 10^{-6}$ N</td>
</tr>
<tr>
<td>Electric attraction between the proton and the electron in a hydrogen atom</td>
<td>$8.2 \times 10^{-8}$ N</td>
</tr>
<tr>
<td>Weight of a very small bacterium</td>
<td>$1 \times 10^{-18}$ N</td>
</tr>
<tr>
<td>Weight of a hydrogen atom</td>
<td>$1.6 \times 10^{-26}$ N</td>
</tr>
<tr>
<td>Weight of an electron</td>
<td>$8.9 \times 10^{-30}$ N</td>
</tr>
<tr>
<td>Gravitational attraction between the proton and the electron in a hydrogen atom</td>
<td>$3.6 \times 10^{-47}$ N</td>
</tr>
</tbody>
</table>

A common instrument for measuring force magnitudes is the *spring balance*. It consists of a coil spring enclosed in a case with a pointer attached to one end. When forces are applied to the ends of the spring, it stretches by an
amount that depends on the force. We can make a scale for the pointer by using a number of identical bodies with weights of exactly 1 N each. When one, two, or more of these are suspended simultaneously from the balance, the total force stretching the spring is 1 N, 2 N, and so on, and we can label the corresponding positions of the pointer 1 N, 2 N, and so on. Then we can use this instrument to measure the magnitude of an unknown force. We can also make a similar instrument that measures pushes instead of pulls.

1.2 Superposition of Forces

When you throw a ball, at least two forces act on it: the push of your hand and the downward pull of gravity. Experiment shows that when two forces \( \vec{F}_1 \) and \( \vec{F}_2 \) act at the same time at the same point on a body (Fig. 1.3), the effect on the body’s motion is the same as if a single force \( \vec{R} \) were acting equal to the vector sum, or resultant, of the original forces: \( \vec{R} = \vec{F}_1 + \vec{F}_2 \). More generally, any number of forces applied at a point on a body have the same effect as a single force equal to the vector sum of the forces. This important principle is called superposition of forces.

Figure 1.3 - Superposition of forces.

Two forces \( \vec{F}_1 \) and \( \vec{F}_2 \) acting on a body at point \( O \) have the same effect as a single force \( \vec{R} \) equal to their vector sum.

That’s why we often describe a force \( \vec{F} \) in terms of its \( x \)- and \( y \)-components \( F_x \) and \( F_y \). Note that the \( x \)- and \( y \)-coordinate axes do not have to be horizontal and vertical, respectively. As an example, Fig. 1.4 shows a crate being pulled up a ramp by a force \( \vec{F} \). In this situation it’s most convenient to choose one axis to
be parallel to the ramp and the other to be perpendicular to the ramp. For the case shown in Fig. 1.4, both $F_x$ and $F_y$ are positive; in other situations, depending on your choice of axes and the orientation of the force $\vec{F}$, either $F_x$ or $F_y$ may be negative or zero.

![Figure 1.4 - $F_x$ and $F_y$ are the components of $\vec{F}$ parallel and perpendicular to the sloping surface of the inclined plane](image)

Otherwise, the diagram would include the same force twice. We will draw such a wiggly line in any force diagram where a force is replaced by its components. Look for this wiggly line in other figures in this and subsequent chapters.

We will often need to find the vector sum (resultant) of all forces acting on a body. We call this the net force acting on the body. We will use the Greek letter $\sum$ (capital sigma, equivalent to the Roman S) as a shorthand notation for a sum. If the forces are labeled $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, and so on, we can write.

$$\vec{R} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$$

$$\vec{F} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$$

We read $\sum \vec{F}$ as “the vector sum of the forces” or “the net force.” The $x$-component of the net force is the sum of the $x$-components of the individual forces, and likewise for the $y$-component (Fig. 1.5):
\[ R_x = \sum F_x, \quad R_y = \sum F_y \]  

(1.2)

Each component may be positive or negative, so be careful with signs when you evaluate these sums.

Once we have \( R_x \) and \( R_y \) we can find the magnitude and direction of the net force \( \vec{R} = \sum \vec{F} \) acting on the body. The magnitude is

\[ R = \sqrt{R_x^2 + R_y^2} \]

and the angle \( \theta \) between \( \vec{R} \) and the +x-axis can be found from the relationship \( \tan \theta = R_y/R_x \). The components \( R_x \) and \( R_y \) may be positive, negative, or zero, and the angle \( \theta \) may be in any of the four quadrants.

In three-dimensional problems, forces may also have \( z \)-components; then we add the equation \( R_z = \sum F_z \) to Eq. (1.2). The magnitude of the net force is then

\[ R = \left( R_x^2 + R_y^2 + R_z^2 \right)^{1/2} \]

1.3 Newton’s First Law

How do the forces that act on a body affect its motion? To begin to answer this question, let’s first consider what happens when the net force on a body is zero.
You would almost certainly agree that if a body is at rest, and if no net force acts on it (that is, no net push or pull), that body will remain at rest. But what if there is zero net force acting on a body in motion?

To see what happens in this case, suppose you slide a hockey puck along a horizontal tabletop, applying a horizontal force to it with your hand (Fig. 1.6a).

(a) Table: a puck stops short.
(b) Ice: a puck slides farther.
(c) Air-hockey table: a puck slides even farther

Figure 1.6 - The slicker the surface, the farther a puck slides after being given an initial velocity. On an air-hockey table (c) the friction force is practically zero, so the puck continues with almost constant velocity.

After you stop pushing, the puck does not continue to move indefinitely; it slows down and stops. To keep it moving, you have to keep pushing (that is, applying a force). You might come to the “common sense” conclusion that bodies in motion naturally come to rest and that a force is required to sustain motion.

But now imagine pushing the puck across a smooth surface of ice (Fig. 1.6b). After you quit pushing, the puck will slide a lot farther before it stops. Put it on an air-hockey table, where it floats on a thin cushion of air, and it moves still farther (Fig. 1.6c). In each case, what slows the puck down is friction, an interaction between the lower surface of the puck and the surface on which it slides.

Each surface exerts a friction force on the puck that resists the puck’s motion; the difference in the three cases is the magnitude of the friction force.
The ice exerts less friction than the tabletop, so the puck travels farther. The gas molecules of the air-hockey table exert the least friction of all. If we could eliminate friction completely, the puck would never slow down, and we would need no force at all to keep the puck moving once it had been started. Thus the “common sense” idea that a force is required to sustain motion is incorrect.

Experiments like the ones we’ve just described show that when no net force acts on a body, the body either remains at rest or moves with constant velocity in a straight line. Once a body has been set in motion, no net force is needed to keep it moving. We call this observation the Newton’s first law of motion:

The Newton’s first law of motion is as follows: A body acted on by no net force has a constant velocity (which may be zero) and zero acceleration.

The tendency of a body to keep moving once it is set in motion is called inertia. You use inertia when you try to get ketchup out of a bottle by shaking it. First you start the bottle (and the ketchup inside) moving forward; when you jerk the bottle back, the ketchup tends to keep moving forward and, you hope, ends up on your burger. Inertia is also the tendency of a body at rest to remain at rest. You may have seen a tablecloth yanked out from under the china without breaking anything. The force on the china isn’t great enough to make it move appreciably during the short time it takes to pull the tablecloth away.

It’s important to note that the net force is what matters in Newton’s first law. For example, a physics book at rest on a horizontal tabletop has two forces acting on it: an upward supporting force, or normal force, exerted by the tabletop (see Fig. 1.2a) and the downward force of the earth’s gravity (which acts even if the tabletop is elevated above the ground; see Fig. 1.2d). The upward push of the surface is just as great as the downward pull of gravity, so the net force acting on the book (that is, the vector sum of the two forces) is zero. In agreement with Newton’s first law, if the book is at rest on the tabletop, it remains at rest. The same principle applies to a hockey puck sliding on a
horizontal, frictionless surface: The vector sum of the upward push of the surface and the downward pull of gravity is zero. Once the puck is in motion, it continues to move with constant velocity because the net force acting on it is zero.

Here’s another example. Suppose a hockey puck rests on a horizontal surface with negligible friction, such as an air-hockey table or a slab of wet ice. If the puck is initially at rest and a single horizontal force $\vec{F}_1$ acts on it (Fig. 1.7a), the puck starts to move. If the puck is in motion to begin with, the force changes its speed, its direction, or both, depending on the direction of the force. In this case the net force is equal to $\vec{F}_1$, which is not zero. (There are also two vertical forces: the earth’s gravitational attraction and the upward normal force exerted by the surface. But as we mentioned earlier, these two forces cancel.)

Now suppose we apply a second force, $\vec{F}_2$ (Fig. 1.7b), equal in magnitude to $\vec{F}_1$ but opposite in direction. The two forces are negatives of each other, $\vec{F}_1 = -\vec{F}_2$ and their vector sum is zero:

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 = \vec{F}_1 - \vec{F}_1 = 0.$$

Again, we find that if the body is at rest at the start, it remains at rest; if it is initially moving, it continues to move in the same direction with constant speed.

These results show that in Newton’s first law, zero net force is equivalent to no force at all.

When a body is either at rest or moving with constant velocity (in a straight line with constant speed), we say that the body is in equilibrium. For a body to be in equilibrium, it must be acted on by no forces, or by several forces such that their vector sum, that is, the net force is zero:
(a) A puck on a frictionless surface accelerates when acted on by a single horizontal force.

(b) This puck is acted on by two horizontal forces whose vector sum is zero. The puck behaves as though no forces act on it.

Figure 1.7 - (a) A hockey puck accelerates in the direction of a net applied force \( \sum F \). (b) When the net force is zero, the acceleration is zero, and the puck is in equilibrium

\[
\sum F = 0. 
\]  

(1.3)

We’re assuming that the body can be represented adequately as a point particle. When the body has finite size, we also have to consider where on the body the forces are applied.

1.4 Newton’s Second Law

Newton’s first law tells us that when a body is acted on by zero net force, the body moves with constant velocity and zero acceleration. In Fig. 1.8a, a hockey puck is sliding to the right on wet ice. There is negligible friction, so there are no horizontal forces acting on the puck; the downward force of gravity and the upward normal force exerted by the ice surface sum to zero. So the net force \( \sum F \) acting on the puck is zero, the puck has zero acceleration, and its velocity is constant.

But what happens when the net force is not zero? In Fig. 1.8b we apply a constant horizontal force to a sliding puck in the same direction that the puck is moving. Then \( \sum F \) is constant and in the same horizontal direction as \( \vec{v} \). We
find that during the time the force is acting, the velocity of the puck changes at a constant rate; that is, the puck moves with constant acceleration. The speed of the puck increases, so the acceleration \( \vec{a} \) is in the same direction as \( \vec{\dot{v}} \) and \( \sum \vec{F} \).

In Fig. 1.8c we reverse the direction of the force on the puck so that \( \sum \vec{F} \) acts opposite to \( \vec{\dot{v}} \). In this case as well, the puck has an acceleration; the puck moves more and more slowly to the right. The acceleration \( \vec{a} \) in this case is to the left, in the same direction as \( \sum \vec{F} \). As in the previous case, experiment shows that the acceleration is constant if \( \sum \vec{F} \) is constant.

(a) If there is zero net force on the puck, so \( \sum \vec{F} = 0, \ldots \)

(b) If a constant net force \( \sum \vec{F} \) acts on the puck in the direction of its motion \( \ldots \)

(c) If a constant net force \( g \vec{F} \) acts on the puck opposite to the direction of its motion \( \ldots \)

Figure 1.8 - Using a hockey puck on a frictionless surface to explore the relationship between the force and the acceleration

We conclude that a net force acting on a body causes the body to accelerate in the same direction as the net force. If the magnitude of the net force is constant, as in Figs. 1.8b and 1.8c, then so is the magnitude of the acceleration. These conclusions about net force and acceleration also apply to a body moving along a curved path. For example, Fig. 1.9 shows a hockey puck moving in a horizontal circle on an ice surface of negligible friction. A rope is
attached to the puck and to a stick in the ice, and this rope exerts an inward tension force of constant magnitude on the puck. The net force and acceleration are both constant in magnitude and directed toward the center of the circle.

**Figure 1.9a** shows another experiment to explore the relationship between acceleration and net force. As in Figs. 1.9b and 1.9c, this horizontal force equals the net force on the puck. If we change the magnitude of the net force, the acceleration changes in the same proportion. Doubling the net force doubles the acceleration (Fig. 1.10b), halving the net force halves the acceleration (Fig. 1.10c), and so on. Many such experiments show that for any given body, the magnitude of the acceleration is directly proportional to the magnitude of the net force acting on the body.

![Figure 1.9](image1.png)

**Figure 1.9** - A top view of a hockey puck in uniform circular motion on a frictionless horizontal surface.

![Figure 1.10](image2.png)

**Figure 1.10** - The magnitude of a body’s acceleration $\vec{a}$ is directly proportional to the magnitude of the net force $\sum \vec{F}$ acting on the body of mass $m$.

(a) A constant net force $\sum \vec{F}$ causes a constant acceleration $\vec{a}$.

(b) Doubling the net force doubles the acceleration.

(c) Halving the force halves the acceleration.
1.5 Mass and Force

Our results mean that for a given body, the ratio of the magnitude $|\sum \vec{F}|$ of the net force to the magnitude $a = |\vec{a}|$ of the acceleration is constant, regardless of the magnitude of the net force. We call this ratio the inertial mass, or simply the mass, of the body and denote it by $m$. That is,

$$m = \frac{|\sum \vec{F}|}{a}$$

or

$$|\sum \vec{F}| = ma \quad \text{or} \quad a = \frac{|\sum \vec{F}|}{m}. \quad (1.4)$$

The last of the equations in Eqs. (1.4) says that the greater a body’s mass, the more the body “resists” being accelerated. When you hold a piece of fruit in your hand at the supermarket and move it slightly up and down to estimate its heft, you’re applying a force and seeing how much the fruit accelerates up and down in response. If a force causes a large acceleration, the fruit has a small mass; if the same force causes only a small acceleration, the fruit has a large mass. In the same way, if you hit a table-tennis ball and then a basketball with the same force, the basketball has much smaller acceleration because it has much greater mass.

The SI unit of mass is the kilogram. We can use this standard kilogram, along with Eqs. (1.4), to define the newton:

**One newton is the amount of net force that gives an acceleration of 1 meter per second squared to a body with a mass of 1 kilogram.**

This definition allows us to calibrate the spring balances and other instruments used to measure forces. Because of the way we have defined the newton, it is related to the units of mass, length, and time. For Eqs. (1.4) to be dimensionally consistent, it must be true that

$$1 \text{ newton} = (1 \text{ kilogram}) (1 \text{ meter per second squared})$$
or

\[ 1N = 1kg \cdot \frac{m}{s^2}. \]

We will use this relationship many times in the next few chapters, so keep it in mind.

We can also use Eqs. (1.4) to compare a mass with the standard mass and thus to measure masses. Suppose we apply a constant net force \( \sum \vec{F} \) to a body having a known mass \( m_1 \) and we find an acceleration of magnitude \( a_1 \) (Fig. 1.11a). We then apply the same force to another body having an unknown mass \( m_2 \), and we find an acceleration of magnitude \( a_2 \) (Fig. 1.11b). Then, according to Eqs. (1.4), \( m_1 a_1 = m_2 a_2 \),

\[ \frac{m_2}{m_1} = \frac{a_1}{a_2}. \]  

(same net force)  

(1.5)

(a) A known force \( \sum \vec{F} \) causes an object with mass \( m_1 \) to have an acceleration of magnitude \( a_1 \).

(b) Applying the same force \( \sum \vec{F} \) to a second object and noting the acceleration allows measuring the mass.

(c) When the two objects are fastened together, the same method shows that their composite mass is the sum of their individual masses.

Figure 1.11- For a given net force \( \sum \vec{F} \) acting on a body, the acceleration is inversely proportional to the mass of the body. Masses add like ordinary scalars.
For the same net force, the ratio of the masses of two bodies is the inverse of the ratio of their accelerations. In principle we could use Eq. (1.5) to measure an unknown mass $m_2$, but it is usually easier to determine mass indirectly by measuring the body’s weight.

When two bodies with masses $m_1$ and $m_2$ are fastened together, we find that the mass of the composite body is always $m_1 + m_2$ (Fig. 1.11c).

This additive property of mass may seem obvious, but it has to be verified experimentally. Ultimately, the mass of a body is related to the number of protons, electrons, and neutrons it contains. This wouldn’t be a good way to define mass because there is no practical way to count these particles. But the concept of mass is the most fundamental way to characterize the quantity of matter in a body.

### 1.6 Stating Newton’s Second Law

Experiment shows that the net force on a body is what causes that body to accelerate. If a combination of forces $F_1, F_2, F_3$, and so on is applied to a body, the body will have the same acceleration vector $\sum \vec{F}$ as when only a single force is applied, if that single force is equal to the vector sum $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$ In other words, the principle of superposition of forces (see Fig. 1.4) also holds true when the net force is not zero and the body is accelerating.

Equations (1.4) relate the magnitude of the net force on a body to the magnitude of the acceleration that it produces. We have also seen that the direction of the net force is the same as the direction of the acceleration, whether the body’s path is straight or curved. What’s more, the forces that affect a body’s motion are external forces, those exerted on the body by other bodies in its environment.

Newton wrapped up all these results into a single concise statement that we now call Newton’s second law of motion:
Newton’s second law of motion: If a net external force acts on a body, the body accelerates. The direction of acceleration is the same as the direction of the net force. The mass of the body times the acceleration vector of the body equals the net force vector.

\[ \sum \vec{F} = m \vec{a}. \]  \hfill (1.6)

An alternative statement is that the acceleration of a body is equal to the net force acting on the body divided by the body’s mass:

\[ \vec{a} = \frac{\sum \vec{F}}{m}. \]

Newton’s second law is a fundamental law of nature, the basic relationship between force and motion. Most of the remainder of this chapter and all of the next are devoted to learning how to apply this principle in various situations.

Equation (1.6) has many practical applications. You’ve actually been using it all your life to measure your body’s acceleration. In your inner ear, microscopic hair cells sense the magnitude and direction of the force that they must exert to cause small membranes to accelerate along with the rest of your body. By Newton’s second law, the acceleration of the membranes—and hence that of your body as a whole—is proportional to this force and has the same direction. In this way, you can sense the magnitude and direction of your acceleration even with your eyes closed!

1.7 Using Newton’s Second Law

There are at least four aspects of Newton’s second law that deserve special attention. First, Eq. (1.6) is a vector equation. Usually we will use it in component form, with a separate equation for each component of force and the corresponding component of acceleration:

\[ \sum F_x = m a_x, \quad \sum F_y = m a_y, \quad \sum F_z = m a_z. \]  \hfill (1.7)

\[ \sum F_{x,y,z} \rightarrow \text{Each component of net force on a body…} \]
equals body’s mass times the corresponding acceleration component.

This set of component equations is equivalent to the single vector Eq. (1.6).

Second, the statement of Newton’s second law refers to external forces. It’s impossible for a body to affect its own motion by exerting a force on itself; if it were possible, you could lift yourself to the ceiling by pulling up on your belt!

That’s why only external forces are included in the sum \( \sum \vec{F} \) in Eqs. (1.6) and (1.7).

1.8 Mass and Weight

One of the most familiar forces is the weight of a body, which is the gravitational force that the earth exerts on the body. (If you are on another planet, your weight is the gravitational force that planet exerts on you.) Unfortunately, the terms mass and weight are often misused and interchanged in everyday conversation. It is absolutely essential for you to understand clearly the distinctions between these two physical quantities.

Mass characterizes the inertial properties of a body. Mass is what keeps the china on the table when you yank the tablecloth out from under it. The greater the mass, the greater the force needed to cause a given acceleration; this is reflected in Newton’s second law, \( \sum \vec{F} = ma \).

Weight, on the other hand, is a force exerted on a body by the pull of the earth. Mass and weight are related: Bodies that have large mass also have large weight. A large stone is hard to throw because of its large mass, and hard to lift off the ground because of its large weight.

To understand the relationship between mass and weight, note that a freely falling body has an acceleration of magnitude \( g \). Newton’s second law tells us that a force must act to produce this acceleration. If a 1-kg bod falls with an acceleration of \( 9.8 \text{ m/s}^2 \), the required force has magnitude
\[ F = ma = 9.8 \text{ kg} \cdot \text{m/s}^2 = 9.8N. \]

The force that makes the body accelerate downward is its weight. Any body near the surface of the earth that has a mass of 1 kg must have a weight of 9.8 N to give it the acceleration we observe when it is in free fall. More generally,

\[ P = mg. \quad (1.8) \]

\( P \) – Magnitude of weight of a body, \( m \rightarrow \text{Mass of body}, g \rightarrow \text{Magnitude of acceleration due to gravity}. 

Hence the magnitude \( w \) of a body’s weight is directly proportional to its mass \( m \). The weight of a body is a force, a vector quantity, and we can write Eq. (1.8) as a vector equation (Fig. 1.12):

\[ \vec{P} = mg. \quad (1.9) \]

Remember that \( g \) is the magnitude of \( \vec{g} \), the acceleration due to gravity, so \( g \) is always a positive number, by definition. Thus \( w \), given by Eq. (1.8), is the magnitude of the weight and is also always positive.

Figure 1.12 - Relating the mass and weight of a body

- The relationship of mass to weight: \( \vec{P} = mg. \)
- This relationship is the same whether a body is falling or stationary.

Figure 1.12 - Relating the mass and weight of a body

**1.9 Newton’s Third Law**

A force acting on a body is always the result of its interaction with another body, so forces always come in pairs. You can’t pull on a doorknob without the
doorknob pulling back on you. When you kick a football, the forward force that your foot exerts on the ball launches it into its trajectory, but you also feel the force the ball exerts back on your foot.

In each of these cases, the force that you exert on the other body is in the opposite direction to the force that body exerts on you. Experiments show that whenever two bodies interact, the two forces that they exert on each other are always equal in magnitude and opposite in direction. This fact is called Newton’s third law of motion:

Newton’s third law of motion: If body $A$ exerts a force on body $B$ (an “action”), then body $B$ exerts a force on body $A$ (a “reaction”). These two forces have the same magnitude but are opposite in direction. These two forces act on different bodies.

For example, in Fig. 1.13 $\vec{F}_{A\text{ on }B}$ is the force applied by body $A$ (first subscript) on body $B$ (second subscript), and $\vec{F}_{B\text{ on }A}$ is the force applied by body $B$ (first subscript) on body $A$ (second subscript). In equation form,

$$\vec{F}_{A\text{ on }B} = -\vec{F}_{B\text{ on }A}.$$  \hspace{1cm} (1.10)

A and B the two forces have same magnitude but opposite directions. B and A The two forces act on different bodies. It doesn’t matter whether one body is inanimate (like the soccer ball in Fig. 1.13) and the other is not (like the kicker’s foot): They necessarily exert forces on each other that obey Eq. (1.10).

The two forces have same magnitude but opposite directions: $\vec{F}_{A\text{ on }B} = -\vec{F}_{B\text{ on }A}$

![Figure 1.13 - Newton’s third law of motion](image)
In the statement of Newton’s third law, “action” and “reaction” are the two opposite forces (in Fig. 1.13, $\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}$); we sometimes refer to them as an action–reaction pair. This is not meant to imply any cause-and-effect elation; we can consider either force as the “action” and the other as the “reaction. We often say simply that the forces are “equal and opposite,” meaning that they have equal magnitudes and opposite directions’.

In Fig. 1.13 the action and reaction forces are contact forces that are present only when the two bodies are touching. But Newton’s third law also applies to long-range forces that do not require physical contact, such as the force of gravitational attraction. A table-tennis ball exerts an upward gravitational force on the earth that’s equal in magnitude to the downward gravitational force the earth exerts on the ball. When you drop the ball, both the ball and the earth accelerate toward each other. The net force on each body has the same magnitude, but the earth’s acceleration is microscopically small because its mass is so great. Nevertheless, it does move!

1.10 Free-Body Diagrams

Newton’s three laws of motion contain all the basic principles we need to solve a wide variety of problems in mechanics. These laws are very simple in form, but the process of applying them to specific situations can pose real challenges. In this brief section we’ll point out three key ideas and techniques to use in any problems involving Newton’s laws. You’ll learn others in Chapter 5, which also extends the use of Newton’s laws to cover more complex situations.

1. Newton’s first and second laws apply to a specific body. Whenever you use Newton’s first law, $\vec{F} = 0$, for an equilibrium situation or Newton’s second law, $\sum \vec{F} = m\vec{a}$, for a non-equilibrium situation, you must decide at the beginning to which body you are referring. This decision may sound trivial, but it isn’t.
2. Only forces acting on the body matter. The sum \( \sum \vec{F} \) includes all the forces that act on the body in question. Hence, once you’ve chosen the body to analyze, you have to identify all the forces acting on it. Don’t confuse the forces acting on a body with the forces exerted by that body on some other body. For example, to analyze a person walking, you would include in \( \sum \vec{F} \) the force that the ground exerts on the person as he walks, but not the force that the person exerts on the ground. These forces form an action–reaction pair and are related by Newton’s third law, but only the member of the pair that acts on the body you’re working with goes into \( \sum \vec{F} \).

3. Free-body diagrams are essential to help identify the relevant forces. A free-body diagram shows the chosen body by itself, “free” of its surroundings, with vectors drawn to show the magnitudes and directions of all the forces that act on the body. Be careful to include all the forces acting on the body, but be equally careful not to include any forces that the body exerts on any other body. In particular, the two forces in an action–reaction pair must never appear in the same free-body diagram because they never act on the same body. Furthermore, never include force that a body exerts on itself, since these can’t affect the body’s motion.

2. Applying Newton’s Laws

2.1 Using Newton’s First Law: Particles in Equilibrium

We’ll begin with equilibrium problems, in which we analyze the forces that act on a body that is at rest or moving with constant velocity. We’ll then consider bodies that are not in equilibrium, for which we’ll have to deal with the relationship between forces and motion. We’ll learn how to describe and analyze the contact force that acts on a body when it rests on or slides over a surface. We’ll also analyze the forces that act on a body that moves in a circle with constant speed.
We learned that a body is in equilibrium when it is at rest or moving with constant velocity in an inertial frame of reference. A hanging lamp, a kitchen table, an airplane flying straight and level at a constant speed—all are examples of equilibrium situations. In this section we consider only the equilibrium of a body that can be modeled as a particle. The essential physical principle is Newton’s first law:

$$\sum \vec{F} = 0.$$ \hspace{1cm} (2.1)

**Newton’s first law:** Net force on a body must be zero for a **body in equilibrium**.

$$\sum F_x = 0,$$ Sum of $x$-components of force on body must be zero.

$$\sum F_y = 0,$$ Sum of $y$-components of force on body must be zero.

This section is about using Newton’s first law to solve problems dealing with bodies in equilibrium. Some of these problems may seem complicated, but remember that all problems involving particles in equilibrium are done in the same way. Problem-Solving Strategy 2.1 details the steps you need to follow for any and all such problems. Study this strategy carefully, look at how it’s applied in the worked-out examples, and try to apply it when you solve assigned problems.

**2.2 Using Newton’s Second Law: Dynamics of Particles**

We are now ready to discuss dynamics problems. In these problems, we apply Newton’s second law to bodies on which the net force is not zero. These bodies are not in equilibrium and hence are accelerating:

$$\sum \vec{F} = m\vec{a}.$$ \hspace{1cm} (2.2)

When a passenger with mass $m$ rides in an elevator with $y$-acceleration $a_y$, a scale shows the passenger’s apparent weight to be

$$P = m\left(g + a_y\right).$$
When the elevator is accelerating upward, $a_y$ is positive and $n$ is greater than the passenger’s weight $P = mg$. When the elevator is accelerating downward, $a_y$ is negative and $n$ is less than the weight. If the passenger doesn’t know the elevator is accelerating, she may feel as though her weight is changing; indeed, this is just what the scale shows.

The extreme case occurs when the elevator has a downward acceleration $a_y = -g$—that is, when it is in free fall. In that case $P = 0$ and the passenger seems to be weightless. Similarly, an astronaut orbiting the earth with a spacecraft experiences apparent weightlessness (Fig. 2.1). In each case, the person is not truly weightless because a gravitational force still acts. But the person’s sensations in this free-fall condition are exactly the same as though the person were in outer space with no gravitational force at all. In both cases the person and the vehicle (elevator or spacecraft) fall together with the same acceleration $g$, so nothing pushes the person against the floor or walls of the vehicle.

2.3 Friction Forces. Kinetic and Static Friction

We’ve seen several problems in which a body rests or slides on a surface that exerts forces on the body. Whenever two bodies interact by direct contact (touching) of their surfaces, we describe the interaction in terms of contact forces. The normal force is one example of a contact force; in this section we’ll look in detail at another contact force, the force of friction.

Friction is important in many aspects of everyday life. The oil in a car engine minimizes friction between moving parts, but without friction between
the tires and the road we couldn’t drive or turn the car. Air drag—the friction force exerted by the air on a body moving through it—decreases automotive fuel economy but makes parachutes work. Without friction, nails would pull out and most forms of animal locomotion would be impossible.

First, when a body rests or slides on a surface, we can think of the surface as exerting a single contact force on the body, with force components perpendicular and parallel to the surface (Fig. 2.2). The perpendicular component vector is the normal force, denoted by \( \vec{N} \). The component vector parallel to the surface (and perpendicular to \( \vec{N} \)) is the friction force, denoted by \( \vec{F} \). If the surface is frictionless, then \( \vec{F} \) is zero but there is still a normal force. (Frictionless surfaces are an unattainable idealization, like a massless rope. But we can approximate a surface as frictionless if the effects of friction are negligibly small.) The direction of the friction force is always such as to oppose relative motion of the two surfaces.

The kind of friction that acts when a body slides over a surface is called a kinetic friction force \( \vec{F}_k \). The adjective “kinetic” and the subscript “k” remind us that the two surfaces are moving relative to each other. The magnitude of the kinetic friction force usually increases when the normal force increases. This is why it takes more force to slide a full box of books across the floor than an empty one. Automotive brakes use the same principle: The harder the brake pads are squeezed against the rotating brake discs, the greater the braking effect. In many cases the magnitude of the kinetic friction force \( \vec{F}_k \) is found experimentally to be approximately proportional to the magnitude \( N \) of the normal force:

\[
F_k = \mu N. \tag{2.3}
\]

Where \( F_k \) is the magnitude of kinetic friction force, \( \mu_k \) -is the coefficient of kinetic friction, \( N \) is the magnitude of normal force.
When a block is pushed or pulled over a surface, the surface exerts a contact force on it.

Here $\mu_k$ (pronounced “mu-sub-k”) is a constant called the **coefficient of kinetic friction**. The more slippery the surface, the smaller this coefficient. Because it is a quotient of two force magnitudes, $\mu_k$ is a pure number without units.

**Table 2.1 - Approximate coefficient of kinetic friction**

<table>
<thead>
<tr>
<th>Materials</th>
<th>Coefficient of Static Friction, $\mu_s$</th>
<th>Coefficient of Kinetic Friction, $\mu_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel on steel</td>
<td>0.74</td>
<td>0.57</td>
</tr>
<tr>
<td>Aluminum on steel</td>
<td>0.61</td>
<td>0.47</td>
</tr>
<tr>
<td>Copper on steel</td>
<td>0.53</td>
<td>0.36</td>
</tr>
<tr>
<td>Brass on steel</td>
<td>0.51</td>
<td>0.44</td>
</tr>
<tr>
<td>Zinc on cast iron</td>
<td>0.85</td>
<td>0.21</td>
</tr>
<tr>
<td>Copper on cast iron</td>
<td>1.05</td>
<td>0.29</td>
</tr>
<tr>
<td>Glass on glass</td>
<td>0.94</td>
<td>0.40</td>
</tr>
<tr>
<td>Copper on glass</td>
<td>0.68</td>
<td>0.53</td>
</tr>
<tr>
<td>Teflon on Teflon</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>Teflon on steel</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>Rubber on concrete (dry)</td>
<td>1.0</td>
<td>0.8</td>
</tr>
<tr>
<td>Rubber on concrete (wet)</td>
<td>0.30</td>
<td>0.25</td>
</tr>
</tbody>
</table>

**Table 2.1** lists some representative values of $\mu_k$. Although these values are given with two significant figures, they are only approximate, since friction forces can also depend on the speed of the body relative to the surface. For now we’ll ignore this effect and assume that $\mu_k$ and $F_i$ are independent of speed, in order to concentrate on the simplest cases. Table 2.1 also lists coefficients of static friction.

Friction forces may also act when there is no relative motion. If you try to slide a box across the floor, the box may not move at all because the floor exerts
an equal and opposite friction force on the box. This is called **a static friction force** \( \vec{F}_s \).

In a particular situation, the actual force of static friction can have any magnitude between zero (when there is no other force parallel to the surface) and a maximum value given by \( \mu_s n \):

\[
F_s \leq (F_s)_{\text{max}} = \mu_s N.
\]  

(2.4)

\( F_s \) - Magnitude of static friction force, \( (F_s)_{\text{max}} \) – Maximum static friction force, \( \mu_s \) – Coefficient of static friction, \( n \) – Magnitude of normal force.

Like Eq. (2.3), this is a relationship between magnitudes, not a vector relationship. The equality sign holds only when the applied force \( T \) has reached the critical value at which motion is about to start (Fig. 2.19c). When \( T \) is less than this value (Fig. 2.6b), the inequality sign holds. In that case we have to use the equilibrium conditions \( \sum F = 0 \) to find \( F_s \). If there is no applied force \( (T = 0) \) as in Fig. 2.6a, then there is no static friction force either \( (F_s = 0) \).

In some situations the surfaces will alternately stick (static friction) and slip (kinetic friction). This is what causes the horrible sound made by chalk held at the wrong angle on a blackboard and the shriek of tires sliding on asphalt.
pavement. A more positive example is the motion of a violin bow against the string.

In the linear air tracks used in physics laboratories, gliders move with very little friction because they are supported on a layer of air. The friction force is velocity dependent, but at typical speeds the effective coefficient of friction is of the order of 0.001.

2.4 Rolling Friction. Fluid Resistance and Terminal Speed

Sticking your hand out the window of a fast-moving car will convince you of the existence of fluid resistance, the force that a fluid (a gas or liquid) exerts on a body moving through it. The moving body exerts a force on the fluid to push it out of the way. By Newton’s third law, the fluid pushes back on the body with an equal and opposite force.

The direction of the fluid resistance force acting on a body is always opposite the direction of the body’s velocity relative to the fluid. The magnitude of the fluid resistance force usually increases with the speed of the body through the fluid. This is very different from the kinetic friction force between two surfaces in contact, which we can usually regard as independent of speed. For small objects moving at very low speeds, the magnitude \( F \) of the fluid resistance force is approximately proportional to the body’s speed \( \vartheta \):

\[
F = k \vartheta, \tag{2.5}
\]

where \( k \) is a proportionality constant that depends on the shape and size of the body and the properties of the fluid. Equation (2.5) is appropriate for dust particles falling in air or a ball bearing falling in oil. For larger objects moving through air at the speed of a tossed tennis ball or faster, the resisting force is approximately proportional to \( \vartheta^2 \) rather than to \( \vartheta \). It is then called air drag or simply drag. Airplanes, falling raindrops, and bicyclists all experience air drag. In this case we replace Eq. (2.5) by

\[
F = D\vartheta^2. \tag{2.6}
\]
Because of the $\vartheta^2$ dependence, air drag increases rapidly with increasing speed. The air drag on a typical car is negligible at low speeds but comparable to or greater than rolling resistance at highway speeds. The value of $D$ depends on the shape and size of the body and on the density of the air. You should verify that the units of the constant $k$ in Eq. (2.5) are $N \cdot s^2/m$ or $kg/s$, and that the units of the constant $D$ in Eq. (2.6) are $N \cdot s^2/m^2$ or $kg/m$.

Figure 2.7b shows the free-body diagram. We take the positive y-direction to be downward and neglect any force associated with buoyancy in the oil. Since the ball is moving downward, its speed $\vartheta$ is equal to its $y$-velocity $\vartheta_y$ and the fluid resistance force is in the $-y$-direction. There are no $x$-components, so Newton’s second law gives

$$\sum F_y = mg + (-k \vartheta_y) = ma_y. \quad (2.7)$$

When the ball first starts to move, $\vartheta_y = 0$, the resisting force is zero and the initial acceleration is $a_y = g$. As the speed increases, the resisting force also increases, until finally it is equal in magnitude to the weight. At this time $mg - k \vartheta_y = 0$, the acceleration is zero, and there is no further increase in speed. The final speed $\vartheta_f$, called the terminal speed, is given by $mg - k \vartheta_f = 0$, or (terminal speed, fluid resistance $f = k \vartheta$)

$$\vartheta_f = \frac{mg}{k} \quad (2.8)$$

**Figure 2.4** shows how the acceleration, velocity, and position vary with time. As time goes by, the acceleration approaches zero and the velocity approaches $\vartheta_f$ (remember that we chose the positive $y$-direction to be down). The slope of the graph of $y$ versus $t$ becomes constant as the velocity becomes constant. To see how the graphs in Fig. 2.4 are derived, we must find the relationship between velocity and time during the interval before the terminal
speed is reached. We go back to Newton’s second law for the falling ball, Eq. (2.7), which we rewrite with \( a_y = \frac{d\theta_y}{dt} \):

\[
\begin{align*}
\frac{dy}{dt} &= \frac{d\theta_y}{dt} \\
\frac{d\theta_y}{dt} &= \text{No fluid resistance: constant acceleration.} \\
\frac{d\theta_y}{dt} &= \text{With fluid resistance: acceleration decreases.}
\end{align*}
\]

Figure 2.4 – Graphs the motion of a body falling without fluid resistance and with fluid resistance proportional to the speed
\[ m \frac{d \vartheta_y}{dt} = mg - k \vartheta_y. \]

After rearranging terms and replacing \( mg/k \) by \( \vartheta_t \), we integrate both sides, noting that \( \vartheta_y = 0 \) when \( t = 0 \):

\[
\int_0^\vartheta \frac{d \vartheta_y}{\vartheta_y - \vartheta_t} = -\frac{k}{m} \int_0^t dt,
\]

which integrates to

\[
\ln \frac{\vartheta_y - \vartheta_t}{\vartheta_t} = -\frac{k}{m} t, \quad \text{or} \quad 1 - \frac{\vartheta_y}{\vartheta_t} = e^{-(k/m)t},
\]

and finally

\[
\vartheta_y = \vartheta_t \left[1 - e^{-(k/m)t}\right]. \tag{2.9}
\]

Note that \( \vartheta_y \) becomes equal to the terminal speed \( \vartheta_y \) only in the limit that \( t \to \infty \); the ball cannot attain terminal speed in any finite length of time.

The derivative of \( \vartheta_y \) in Eq. (2.9) gives \( a_y \) as a function of time, and the integral of \( \vartheta_y \) gives \( y \) as a function of time. We leave the derivations for you to complete; the results are

\[
a_y = ge^{-(k/m)t}, \tag{2.10}
\]

\[
y = \vartheta_t \left[t - \frac{m}{k} (1 - e^{-(k/m)t})\right]. \tag{2.11}
\]

Now look again at Fig. 2.4, which shows graphs of these three relationships. In deriving the terminal speed in Eq. (2.8), we assumed that the fluid resistance force is proportional to the speed. For an object falling through the air at high speeds, so that the fluid resistance is equal to \( D\vartheta^2 \) as in speed is reached when \( D\vartheta^2 \) equals the weight \( mg \). You can show that the terminal speed \( \vartheta_t \) is given by

\[
\vartheta_t = \sqrt{\frac{mg}{D}} \text{ (terminal speed, fluid resistance } F = D\vartheta^2). \tag{2.12}
\]
This expression for terminal speed explains why heavy objects in air tend to fall faster than light objects. Two objects that have the same physical size but different mass (say, a table-tennis ball and a lead ball with the same radius) have the same value of $D$ but different values of $m$. The more massive object has a higher terminal speed and falls faster. The same idea explains why a sheet of paper falls faster if you first crumple it into a ball; the mass $m$ is the same, but the smaller size makes $D$ smaller (less air drag for a given speed) and $g$, larger.

2.5 Dynamics of Circular Motion

We showed that when a particle moves in a circular path with constant speed, the particle’s acceleration has a constant magnitude $a_{rad}$ given by

$$a_{rad} = \frac{g^2}{R},$$

(2.13)

Where $a_{rad}$ is magnitude of acceleration of an object in uniform circular motion.

$g^2$ is the speed of object, $R$ – Radius of object’s circular path.

The subscript “rad” is a reminder that at each point the acceleration points radially inward toward the center of the circle, perpendicular to the instantaneous velocity.

We can also express the centripetal acceleration $a_{rad}$ in terms of the period $T$, the time for one revolution:

$$T = \frac{2\pi R}{g}.$$  

(2.14)

In terms of the period, $a_{rad}$ is

$$a_{rad} = \frac{4\pi^2 R}{T^2},$$

(2.15)

Where $a_{rad}$ is the magnitude of acceleration of an object in uniform circular motion, $R$ is the radius of object’s circular path, $T^2$ is the period of motion.
Uniform circular motion, like all other motion of a particle, is governed by Newton’s second law. To make the particle accelerate toward the center of the circle, the net force $\sum \vec{F}$ on the particle must always be directed toward the center (Fig. 2.5).

![Figure 2.5 - Net force, acceleration, and velocity in uniform circular motion](image1)

The magnitude of the acceleration is constant, so the magnitude $F_{net}$ of the net force must also be constant. If the inward net force stops acting, the particle flies off in a straight line tangent to the circle (Fig. 2.6).

![Figure 2.6 - What happens if the inward radial force suddenly ceases to act on a body in circular motion](image2)

The magnitude of the radial acceleration is given by $a_{rad} = g^2/R$, so the magnitude $F_{net}$ of the net force on a particle with mass $m$ in uniform circular motion must be

$$F_{net} = ma_{rad} = m \frac{g^2}{R}.$$  

(2.16)

Uniform circular motion can result from any combination of forces, just so the net force $\sum \vec{F}$ is always directed toward the center of the circle and has a constant magnitude. Note that the body need not move around a complete circle: Equation (2.16) is valid for any path that can be regarded as part of a circular arc.
2.6 The Fundamental Forces of Nature

Gravitational interactions include the familiar force of your weight, which results from the earth’s gravitational attraction acting on you. The mutual gravitational attraction of various parts of the earth for each other holds our planet together, and likewise for the other planets (Fig. 2.7a). Newton recognized that the sun’s gravitational attraction for the earth keeps our planet in its nearly circular orbit around the sun.

The second familiar class of forces, electromagnetic interactions, includes electric and magnetic forces. If you run a comb through your hair, the comb ends up with an electric charge; you can use the electric force exerted by this charge to pick up bits of paper. All atoms contain positive and negative electric charge, so atoms and molecules can exert electric forces on one another. Contact forces, including the normal force, friction, and fluid resistance, are the result of electrical interactions between atoms on the surface of an object and atoms in its surroundings (Fig. 2.7b). Magnetic forces, such as those between magnets or between a magnet and a piece of iron, are actually the result of electric charges in motion. For example, an electromagnet causes magnetic interactions because electric charges move through its wires. We will study electromagnetic interactions in detail in the second half of this book.

On the atomic or molecular scale, gravitational forces play no role because electric forces are enormously stronger: The electrical repulsion between two protons is stronger than their gravitational attraction by a factor of about $10^{35}$. But in bodies of astronomical size, positive and negative charges are usually present in nearly equal amounts, and the resulting electrical interactions nearly cancel out. Gravitational interactions are thus the dominant influence in the motion of planets and in the internal structure of stars.

In the 1960s physicists developed a theory that described the electromagnetic and weak interactions as aspects of a single electroweak interaction. This theory has passed every experimental test to which it has been
put. Encouraged by this success, physicists have made similar attempts to
describe the strong, electromagnetic, and weak interactions in terms of a single
grand unified theory (GUT) and have taken steps toward a possible unification
of all interactions into a theory of everything (TOE). Such theories are still
speculative, and there are many unanswered questions in this very active field of
current research.

Figure 2.7 - Examples of the fundamental interactions in nature

a) The gravitational interaction

b) The electromagnetic interaction
3 Work and kinetic energy

3.1 Work

The importance of the energy idea stems from the principle of conservation of energy: Energy is a quantity that can be converted from one form to another but cannot be created or destroyed. In an automobile engine, chemical energy stored in the fuel is converted partially to the energy of the automobile’s motion and partially to thermal energy. In a microwave oven, electromagnetic energy obtained from your power company is converted to thermal energy of the food being cooked. In these and all other processes, the total energy—the sum of all energy present in all different forms—remains the same. No exception has ever been found.

You’d probably agree that it’s hard work to pull a heavy sofa across the room, to lift a stack of encyclopedias from the floor to a high shelf, or to push a stalled car off the road. Indeed, all of these examples agree with the everyday meaning of work—any activity that requires muscular or mental effort. In physics, work has a much more precise definition. By making use of this definition we’ll find that in any motion, no matter how complicated, the total work done on a particle by all forces that act on it equals the change in its kinetic energy—a quantity that’s related to the particle’s mass and speed.

The three examples of work described above—pulling a sofa, lifting encyclopedias, and pushing a car have something in common. In each case you do work by exerting a force on a body while that body moves from one place to another—that is, undergoes a displacement (Fig. 3.1). You do more work if the force is greater (you push harder on the car) or if the displacement is greater (you push the car farther down the road).

The physicist’s definition of work is based on these observations. Consider a body that undergoes a displacement of magnitude \( s \) along a straight line. (For now, we’ll assume that any body we discuss can be treated as a particle so that we can ignore any rotation or changes in shape of the body.)
While the body moves, a constant force $\vec{F}$ acts on it in the same direction as the displacement (Fig. 3.2). We define the work $A$ done by this constant force under these circumstances as the product of the force magnitude $F$ and the displacement magnitude (constant force in direction of straight-line displacement) $S$:

$$A = FS.$$  \hspace{1cm} (3.1)

The work done on the body is greater if either the force $F$ or the displacement $S$ is greater, in agreement with our observations above.

The SI unit of work is the joule (abbreviated J, pronounced “jool,” and named in honor of the 19th-century English physicist James Prescott Joule). From Eq. (3.1) we see that in any system of units, the unit of work is the unit of force multiplied by the unit of distance. In SI units the unit of force is the newton and the unit of distance is the meter, so 1 joule is equivalent to 1 newton-meter ($N \cdot m$):

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}) \quad \text{or} \quad 1 \text{ J} = 1 \text{ N} \cdot \text{m}.$$  

If you lift an object with a weight of 1N (about the weight of a medium-sized apple) a distance of 1 m at a constant speed, you exert a 1-N force on the object in the same direction as its 1-m displacement and so do 1J of work on it.

But it’s important to understand that work can also be negative or zero. This is the essential way in which work as defined in physics differs from the
“everyday” definition of work. When the force has a component in the same direction as the displacement (φ between 0° and 90°), \( \cos \phi \) in Eq. (3.2) is positive and the work \( A \) is positive (Fig. 3.3a). When the force has a component opposite to the displacement (φ between 90° and 180°), \( \cos \phi \) is negative and the work is negative (Fig. 3.3b). When the force is perpendicular to the displacement, \( \phi = 90^\circ \) and the work done by the force is zero (Fig. 3.3c). The cases of zero work and negative work bear closer examination, so let’s look at some examples.

(a) Force \( \vec{F} \) has a component in direction of displacement: \( A = F_x = (F \cos \phi)s \). Work is positive.

(b) Force \( \vec{F} \) has a component opposite to direction of displacement: \( A = F_x = (F \cos \phi)s \). Work is negative (because \( \vec{F} \cos \phi \) is negative for \( 90^\circ < \phi < 180^\circ \)).

(c) Force \( \vec{F} \) (or force component \( F_\perp \)) is perpendicular to direction of displacement:

The force (or force component) does no work on the object.

Figure 3.3 - A constant force \( \vec{F} \) can do positive, negative, or zero work depending on the angle between \( \vec{F} \) and the displacement \( S \)
3.2 Kinetic energy and the work-energy theorem

In Fig. 3.4 a particle with mass \( m \) moves along the \( x \)-axis under the action of a constant net force with magnitude \( \vec{F} \) that points in the positive \( x \)-direction. The particle’s acceleration is constant and given by Newton’s second law: \( F = ma_x \). As the particle moves from point \( x_1 \) to \( x_2 \), it undergoes a displacement \( S = x_2 - x_1 \) and its speed changes from \( \dot{\theta}_1 \) to \( \dot{\theta}_2 \).

\[
\dot{\theta}_2^2 = \dot{\theta}_1^2 + 2a_xS, \\
a_x = \frac{\dot{\theta}_2^2 - \dot{\theta}_1^2}{2S}.
\]

When we multiply this equation by \( m \) and equate \( ma_x \) to the net force \( F \), we find

\[
F = ma_x = m \frac{\dot{\theta}_2^2 - \dot{\theta}_1^2}{2S}.
\]

and

\[
FS = \frac{1}{2} (m\dot{\theta}_2^2 - m\dot{\theta}_1^2).
\]

In Eq. (3.4) the product \( FS \) is the work done by the net force \( F \) and thus is equal to the total work \( A_{\text{tot}} \) done by all the forces acting on the particle. The quantity \( \frac{1}{2}m\dot{\theta}^2 \) is called the \textbf{kinetic energy} \( K \) of the particle:

\[
K = \frac{1}{2}m\dot{\theta}^2,
\]

Where \( K \) is the kinetic energy of a particle, \( m \) is the mass of particle, \( \dot{\theta} \) is the speed of particle.
3.3 Potential energy and energy conservation

We will prove that in some cases the sum of a system’s kinetic and potential energies, called the total mechanical energy of the system, is constant during the motion of the system. This will lead us to the general statement of the law of conservation of energy, one of the most fundamental principles in all of science.

In many situations it seems as though energy has been stored in a system, to be recovered later. For example, you must do work to lift a heavy stone over your head. It seems reasonable that in hoisting the stone into the air you are storing energy in the system, energy that is later converted into kinetic energy when you let the stone fall.

![Figure 3.5](image)

Figure 3.5 - The greater the height of a basketball, the greater the associated gravitational potential energy. As the basketball descends, gravitational potential energy is converted to kinetic energy and the basketball’s speed increases.

This example points to the idea of an energy associated with the position of bodies in a system. This kind of energy is a measure of the potential or possibility for work to be done; if you raise a stone into the air, there is a potential for the gravitational force to do work on it, but only if you allow the stone to fall to the ground. For this reason, energy associated with position is called potential energy. The potential energy associated with a body’s weight and its height above the ground is called gravitational potential energy (Fig. 3.5).

We now have two ways to describe what happens when a body falls without air resistance. The other way is to say that the kinetic energy increases
as the gravitational potential energy decreases. Later in this section we’ll use the work–energy theorem to show that these two descriptions are equivalent.

![Figure 3.6](image)

Let’s derive the expression for gravitational potential energy. Suppose a body with mass $m$ moves along the (vertical) $y$-axis, as in **Fig. 3.6**. The forces acting on it are its weight, with magnitude $P = mg$, and possibly some other forces; we call the vector sum (resultant) of all the other forces $F_{\text{other}}$. We’ll assume that the body stays close enough to the earth’s surface that the weight is constant. We want to find the work done by the weight when the body moves downward from a height $y_1$ above the origin to a lower height $y_2$ (Fig. 3.6a). The weight and displacement are in the same direction, so the work $A_{\text{grav}}$ done on the body by its weight is positive:

$$A_{\text{grav}} = FS = P(y_1 - y_2) = mgy_1 - mgy_2.$$  \hspace{1cm} (3.3)

This expression also gives the correct work when the body moves *upward* and $y_2$ is greater than $y_1$ (Fig. 3.6b). In that case the quantity $y_1 - y_2$ is negative,
and $A_{\text{grav}}$ is negative because the weight and displacement are opposite in direction.

Equation (3.5) shows that we can express $A_{\text{grav}}$ in terms of the values of the quantity $mgy$ at the beginning and end of the displacement. This quantity is called the gravitational potential energy, $E_{\text{grav}}$:

$$E_{\text{grav}} = mgy,$$  \hspace{1cm} (3.4)

Where $E_{\text{grav}}$ is the gravitational potential energy associated with a particle, $m$ is the mass of particle, $g$- acceleration due to gravity, $y$ is the vertical coordinate of particle ($y$ increases if particle moves upward).

Its initial value is $E_{\text{grav},1} = mgy_1$ and its final value is $E_{\text{grav},2} = mgy_2$. The change in $E_{\text{grav}}$ is the final value minus the initial value, or $\Delta E_{\text{grav}} = E_{\text{grav},2} - E_{\text{grav},1}$. Using Eq. (3.4), we can rewrite Eq. (3.3) for the work done by the gravitational force during the displacement from $y_1$ to $y_2$:

$$A_{\text{grav}} = E_{\text{grav},1} - E_{\text{grav},2} = -(E_{\text{grav},2} - E_{\text{grav},1}) = -\Delta E_{\text{grav}}.$$

$$A_{\text{grav}} = mgy_1 - mgy_2 = E_{\text{grav},1} - E_{\text{grav},2} = -\Delta E_{\text{grav}}.$$  \hspace{1cm} (3.5)

The negative sign in front of $\Delta E_{\text{grav}}$ is essential. When the body moves up, $y$ increases, the work done by the gravitational force is negative, and the gravitational potential energy increases ($\Delta E_{\text{grav}} > 0$). When the body moves down, $y$ decreases, the gravitational force does positive work, and the gravitational potential energy decreases ($\Delta E_{\text{grav}} < 0$). It’s like drawing money out of the bank (decreasing $E_{\text{grav}}$) and spending it (doing positive work). The unit of potential energy is the joule (J), the same unit as is used for work.

It is not correct to call $E_{\text{grav}} = mgy$ the “gravitational potential energy of the body.” The reason is that $E_{\text{grav}}$ is a shared property of the body and the earth. The value of $E_{\text{grav}}$ increases if the earth stays fixed and the body moves upward,
away from the earth; it also increases if the body stays fixed and the earth is moved away from it. Notice that the formula \( E_{\text{grav}} = mgy \) involves characteristics of both the body (its mass \( m \)) and the earth (the value of \( g \)).

### 3.4 Elastic potential energy

In many situations we encounter potential energy that is not gravitational in nature. One example is a rubber-band slingshot. Work is done on the rubber band by the force that stretches it, and that work is stored in the rubber band until you let it go. Then the rubber band gives kinetic energy to the projectile.

To keep such an ideal spring stretched by a distance \( x \), we must exert a force \( F = kx \), where \( k \) is the force constant of the spring. Many elastic bodies show this same direct proportionality between force \( F \) and displacement \( x \), provided that \( x \) is sufficiently small.

Let’s proceed just as we did for gravitational potential energy. We begin with the work done by the elastic (spring) force and then combine this with the work–energy theorem. The difference is that gravitational potential energy is a shared property of a body and the earth, but elastic potential energy is stored in just the spring (or other deformable body).

![Diagram of elastic potential energy](image)

**Figure 3.7** - Calculating the work done by a spring attached to a block on a horizontal surface. The quantity \( x \) is the extension or compression of the spring
In Fig. 3.7a the block is at \( x = 0 \) when the spring is neither stretched nor compressed. We move the block to one side, thereby stretching or compressing the spring, then let it go. As the block moves from a different position \( x_1 \) to a different position \( x_2 \), how much work does the elastic (spring) force do on the block?

We found in Section 3.2 that the work we must do on the spring to move one end from an elongation \( x_1 \) to a different elongation \( x_2 \) is

\[
A = \frac{1}{2} k x_2^2 - \frac{1}{2} k x_1^2,
\]

where \( k \) is the force constant of the spring. If we stretch the spring farther, we do positive work on the spring; if we let the spring relax while holding one end, we do negative work on it. This expression for work is also correct when the spring is compressed such that \( x_1, x_2 \), or both are negative. Now, from Newton’s third law the work done by the spring is just the negative of the work done on the spring. So by changing the signs in Eq. (3.6), we find that in a displacement from \( x_1 \) to \( x_2 \) the spring does an amount of work \( A_{el} \) given by

\[
A_{el} = \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2.
\]

The subscript “el” stands for elastic. When both \( x_1 \) and \( x_2 \) are positive and \( x_2 > x_1 \) (Fig. 3.7b), the spring does negative work on the block, which moves in the \(+x\)-direction while the spring pulls on it in the \(-x\)-direction. The spring stretches farther, and the block slows down. When both \( x_1 \) and \( x_2 \) are positive and \( x_2 < x_1 \) (Fig. 3.7c), the spring does positive work as it relaxes and the block speeds up. If the spring can be compressed as well as stretched, \( x_1, x_2 \), or both may be negative, but the expression for \( A_{el} \) is still valid. In Fig. 3.7d, both \( x_1 \) and \( x_2 \) are negative, but \( x_2 \) is less negative than \( x_1 \); the compressed spring does positive work as it relaxes, speeding the block up.

Just as for gravitational work, we can express Eq. (3.7) for the work done by the spring in terms of a quantity at the beginning and end of the
displacement. This quantity is \( \frac{1}{2}kx^2 \), and we define it to be the **elastic potential energy**:

\[
E_{el} = \frac{1}{2}kx^2,
\]

(3.8)

Where \( E_{el} \) is the elastic potential energy, \( k \) is the force constant of spring, \( x^2 \) is the elongation of spring (\( x>0 \) if stretched; \( x<0 \) if compressed).

An important difference between gravitational potential energy \( E_{grav} = mgx \) and elastic potential energy \( E_{el} = \frac{1}{2}kx^2 \) is that we cannot choose \( x = 0 \) to be wherever we wish. In Eq. (3.8), \( x = 0 \) must be the position at which the spring is neither stretched nor compressed. At that position, both its elastic potential energy and the force that it exerts are zero.

### 3.5 Power

The definition of work makes no reference to the passage of time. If you lift a barbell weighing 100 N through a vertical distance of 1.0 m at constant velocity, you do 100 N \( \times \) 1.0 m = 100 J of work whether it takes you 1 second, 1 hour, or 1 year to do it. But often we need to know how quickly work is done. We describe this in terms of **power**. In ordinary conversation the word “power” is often synonymous with “energy” or “force.” In physics we use a much more precise definition: **Power** is the time rate at which work is done. Like work and energy, power is a scalar quantity.

The average work done per unit time, or **average power** \( P_{av} \), is defined to be

\[
P_{av} = \frac{\Delta A}{\Delta t}.
\]

(3.9)

Where \( P_{av} \) is the average power during time interval \( \Delta t \), \( \Delta A \) is the work done during time interval, \( \Delta t \) is the duration of time interval.

The rate at which work is done might not be constant. We define **instantaneous power** \( P \) as the quotient in Eq. (3.9) as \( t \) approaches zero:
\[ P = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}. \tag{3.10} \]

The SI unit of power is the **watt** \((W)\), named for the English inventor James Watt. One watt equals 1 joule per second: \(1 \, W = 1 \, J \, s\). The kilowatt \(1 \, kW = 10^3 \, W\) and the megawatt \(1 \, MW = 10^6 \, W\) are also commonly used.

The watt is a familiar unit of *electrical* power; a 100-W light bulb converts 100 J of electrical energy into light and heat each second. But there’s nothing inherently electrical about a watt. A light bulb could be rated in horsepower, and an engine can be rated in kilowatts.

The average power is

\[ P_{AV} = \frac{F \Delta S}{\Delta t} = F \frac{\Delta S}{\Delta t} = F \theta_{AV}. \tag{6.11} \]

Instantaneous power \(P\) is the limit of this expression as \(t \to 0\):

\[ P = F \dot{\theta}, \tag{6.12} \]

where \(\dot{\theta}\) is the magnitude of the instantaneous velocity. We can also express Eq. (3.12) in terms of the scalar product:

\[ P = F \cdot \dot{\vec{r}}. \]

### 4 Momentum, impulse, and collisions

#### 4.1. Momentum and impulse

Many questions involving forces can’t be answered by directly applying Newton’s second law, \(\vec{F} = m\vec{a}\). For example, when a truck collides head-on with a compact car, what determines which way the wreckage moves after the collision? In playing pool, how do you decide how to aim the cue ball in order to knock the eight ball into the pocket? And when a meteorite collides with the earth, how much of the meteorite’s kinetic energy is released in the impact?

All of these questions involve forces about which we know very little: the forces between the car and the truck, between the two pool balls, or between the
meteorite and the earth. Remarkably, we will find in this chapter that we don’t have to know *anything* about these forces to answer questions of this kind!

Our approach uses two new concepts, momentum and impulse, and a new conservation law, conservation of momentum. This conservation law is every bit as important as the law of conservation of energy. The law of conservation of momentum is valid even in situations in which Newton’s laws are inadequate, such as bodies moving at very high speeds (near the speed of light) or objects on a very small scale (such as the constituents of atoms). Within the domain of Newtonian mechanics, conservation of momentum enables us to analyze many situations that would be very difficult if we tried to use Newton’s laws directly. Among these are collision problems, in which two bodies collide and can exert very large forces on each other for a short time. We’ll also use momentum ideas to solve problems in which an object’s mass changes as it moves, including the important special case of a rocket (which loses mass as it expends fuel).

In Section 3.2 we re-expressed Newton’s second law for a particle, \( \vec{F} = m\vec{a} \), in terms of the work–energy theorem. This theorem helped us tackle a great number of problems and led us to the law of conservation of energy. Let’s return to \( \vec{F} = m\vec{a} \) and see yet another useful way to restate this fundamental law.

Consider a particle of constant mass \( m \). Because \( \vec{a} = d\vec{\mathbf{v}}/dt \), we can write Newton’s second law for this particle as

\[
\sum \vec{F} = m \frac{d\vec{\mathbf{v}}}{dt} = \frac{d}{dt}(m\vec{\mathbf{v}}).
\]

(4.1)

We can move the mass \( m \) inside the derivative because it is constant. Thus Newton’s second law says that the net force \( \sum \vec{F} \) acting on a particle equals the time rate of change of the product of the particle’s mass and velocity. We’ll call this product the momentum, or linear momentum, of the particle:

\[
\vec{p} = m\vec{\mathbf{v}}.
\]

(4.2)
The greater the mass \( m \) and speed \( \mathbf{\dot{b}} \) of a particle, the greater is its magnitude of momentum \( m\mathbf{\dot{b}} \). Keep in mind that momentum is a vector quantity with the same direction as the particle’s velocity (Fig. 4.1). A car driving north at 20 m s\(^{-1}\) and an identical car driving east at 20 m s\(^{-1}\) have the same magnitude of momentum but different momentum vectors because their directions are different.

![Figure 4.1 - Momentum \( \mathbf{\dot{p}} \) is a vector quantity; a particle’s momentum has the same direction as its velocity \( \mathbf{\dot{b}} \)](image)

We often express the momentum of a particle in terms of its components. If the particle has velocity components \( \mathbf{\dot{a}}_x \), \( \mathbf{\dot{a}}_y \), and \( \mathbf{\dot{a}}_z \), then its momentum components \( p_x \), \( p_y \), and \( p_z \) (which we also call the \( x \)-momentum, \( y \)-momentum, and \( z \)-momentum) are

\[
p_x = m\mathbf{\dot{a}}_x, \quad p_y = m\mathbf{\dot{a}}_y, \quad p_z = m\mathbf{\dot{a}}_z.
\]

These three component equations are equivalent to Eq. (4.2).

The units of the magnitude of momentum are units of mass times speed; the SI units of momentum are \( \text{kg m/s} \). The plural of momentum is “momenta.”

Let’s now substitute the definition of momentum, Eq. (4.2), into Eq. (4.1):

\[
\sum \mathbf{\dot{F}} = \frac{d\mathbf{\dot{p}}}{dt}
\]

Where \( \sum \mathbf{\dot{F}} \) = Newton’s second law in terms of momentum; \( \frac{d\mathbf{\dot{p}}}{dt} \) equals the rate of change of the particle’s momentum.

**The net force (vector sum of all forces) acting on a particle equals the time rate of change of momentum of the particle.** This, not \( \mathbf{\dot{F}} = m\mathbf{\dot{a}} \), is the form in which Newton originally stated his second law (although he called
momentum the “quantity of motion”). This law is valid only in inertial frames of reference. As Eq. 4.4 shows, a rapid change in momentum requires a large net force, while a gradual change in momentum requires a smaller net force.

4.1.1 The impulse-momentum theorem

Both a particle’s momentum \( p = m\dot{\theta} \) and its kinetic energy \( K = \frac{1}{2} m\dot{\theta}^2 \) depend on the mass and velocity of the particle. What is the fundamental difference between these two quantities? A purely mathematical answer is that momentum is a vector whose magnitude is proportional to speed, while kinetic energy is a scalar proportional to the speed squared. But to see the physical difference between momentum and kinetic energy, we must first define a quantity closely related to momentum called impulse.

Let’s first consider a particle acted on by a constant net force \( \sum \vec{F} \) during a time interval \( \Delta t \) from \( t_1 \) to \( t_2 \). The impulse of the net force, denoted by \( \vec{J} \), is defined to be the product of the net force and the time interval:

\[
\vec{J} = \sum \vec{F} (t_2 - t_1) = \sum \vec{F} \Delta t \tag{4.5}
\]

Impulse is a vector quantity; its direction is the same as the net force \( \sum \vec{F} \). The SI unit of impulse is the newton-second (NS). Because 1N=1kgm/s\(^2\), an alternative set of units for impulse is kgm the same as for momentum.

To see what impulse is good for, let’s go back to Newton’s second law as restated in terms of momentum, Eq. (4.4). If the net force \( \sum \vec{F} \) is constant, then \( dp/dt \) is also constant. In that case, \( dp/dt \) is equal to the total change in momentum \( \sum \vec{F} = dp / dt \) during the time interval \( t_2 - t_1 \), divided by the interval:

\[
\sum \vec{F} = \frac{(p_2 - p_1)}{t_2 - t_1}.
\]

Multiplying this equation by \( t_2 - t_1 \), we have
\( \Sigma \hat{F} (t_2 - t_1) = (\hat{p}_2 - \hat{p}_1). \)

Comparing with Eq. (4.5), we end up with
\[
\vec{J} = \hat{p}_2 - \hat{p}_1 = \Delta \hat{p}.
\] (4.6)

\( J \) = Impulse of a constant net force; \( p_2 \) = final momentum; \( p_1 \) = initial momentum; \( \Delta p \) = change in momentum.

The impulse-momentum theorem also holds when forces are not constant. To see this, we integrate both sides of Newton's law over time between the limits \( t_1 \) and \( t_2 \):

\[
\int_{t_1}^{t_2} \Sigma \hat{F} \, dt = \int_{t_1}^{t_2} \frac{d\hat{p}}{dt} \, dt = \int_{p_1}^{p_2} \frac{d\hat{p}}{dt} = \hat{p}_2 - \hat{p}_1.
\]

We see from Eq. (4.6) that the integral on the left is the impulse of the net force:
\[
\vec{J} = \int_{t_1}^{t_2} \Sigma \hat{F} \, dt.
\] (4.7)

Where \( J \) is the impulse of general force; \( dt \) is the time integral of net force.

If the net force \( F \) is constant, the integral in Eq(4.7) reduces to Eq.(4.5). We can define an average net force \( \hat{F} \) such that even when \( F \) is not constant, the impulse \( J \) is given by
\[
\vec{J} = \hat{F}_{av} (t_2 - t_1).
\] (4.8)

Figure 4.2a shows the \( x \)-component of net force \( F_x \) as a function of time during a collision. This might represent the force on a soccer ball that is in contact with a player’s foot from time \( t_1 \) to \( t_2 \). The \( x \)-component of impulse during this interval is represented by the red area under the curve between \( t_1 \) and \( t_2 \). This area is equal to the green rectangular area bounded by \( t_1 \), \( t_2 \), and \( F_{av \ x} \), so \( F_{av \ x} \ t_2 - t_1 \) is equal to the impulse of the actual time-varying force during the same interval. Note that a large force acting for a short time can have the same impulse as a smaller force acting for a longer time if the areas under the force–
time curves are the same (Fig. 4.2b). A small force acting for a relatively long time (as when you land with your legs bent) has the same effect as a larger force acting for a short time (as when you land stiff-legged). Automotive air bags use the same principle.

Figure 4.2 - The meaning of the area under a graph of \( F_x \) versus \( t \).

Both impulse and momentum are vector quantities, and Eqs. (4.5)–(4.8) are vector equations. It’s often easiest to use them in component form:

\[
J_x = \int_{t_1}^{t_2} \sum F_x \, dt = (F_{opm})_x (t_2 - t_1) = p_{2x} - p_{1x} = m\dot{q}_{2x} - m\dot{q}_{1x},
\]

\[
J_y = \int_{t_1}^{t_2} \sum F_y \, dt = (F_{opm})_y (t_2 - t_1) = p_{2y} - p_{1y} = m\dot{q}_{2y} - m\dot{q}_{1y}.
\]

**4.1.2 Momentum and kinetic energy compared**

We can now see the fundamental difference between momentum and kinetic energy. The impulse–momentum theorem, \( J = p_2 - p_1 \), says that changes in a particle’s momentum are due to impulse, which depends on the time over which the net force acts. By contrast, the work–energy theorem,

\[
E_{tot} = K_2 - K_1,
\]
tells us that kinetic energy changes when work is done on a particle; the total work depends on the distance over which the net force acts.

Let’s consider a particle that starts from rest at \( t_1 \) so that \( \vec{a}_1 = 0 \). Its initial momentum is \( \vec{p}_1 = m \vec{a}_1 = 0 \) and its initial kinetic energy is \( K_1 = \frac{1}{2} m \vec{a}_1^2 \). Now let a constant net force equal to \( \vec{F} \) act on that particle from time until time. During this interval, the particle moves a distance \( S \) in the direction of the force. From Eq.(4.6), the particle’s momentum at time is

\[
\vec{p}_2 = \vec{p}_1 + \vec{J} = \vec{J},
\]

where \( \vec{J} = F (t_2 - t_1) \) is the impulse that acts on the particle. So the omentum of a particle equals the impulse that accelerated it from rest to its present speed; impulse is the product of the net force that accelerated the particle and the time required for the acceleration. By comparison, the kinetic energy of the particle at \( t_2 \) is \( K_2 = A_{tot} = F S \), the total work done on the particle to accelerate it from rest. The total work is the product of the net force and the distance required to accelerate the particle.

Both the impulse–momentum and work–energy theorems rest on the foundation of Newton’s laws. They are integral principles, relating the motion at two different times separated by a finite interval. By contrast, Newton’s second law itself (in either of the forms \( \vec{F} = m \vec{a} \) or \( \vec{F} = \frac{d\vec{p}}{dt} \)) is a differential principle that concerns the rate of change of velocity or momentum at each instant.

**4.2 Conservation of momentum**

The concept of momentum is particularly important in situations in which we have two or more bodies that interact. To see why, let’s consider first an idealized system of two bodies that interact with each other but not with anything else— for example, two astronauts who touch each other as they float freely in the zero-gravity environment of outer space (Fig. 4.3). Think of the
astronauts as particles. Each particle exerts a force on the other; according to Newton’s third law, the two forces are always equal in magnitude and opposite in direction. Hence, the *impulses* that act on the two particles are equal in magnitude and opposite in direction, as are the changes in momentum of the two particles.

Figure 4.3 - Two astronauts push each other as they float freely in the zero-gravity environment of space

Let’s go over that again with some new terminology. For any system, the forces that the particles of the system exert on each other are called **internal forces**. Forces exerted on any part of the system by some object outside it are called **external forces**. For the system shown in Fig. 4.3, the internal forces are $F_{B \text{ on } A}$, exerted by particle $B$ on particle $A$, and $F_{A \text{ on } B}$, exerted by particle $A$ on particle $B$. There are no external forces; when this is the case, we have an **isolated system**.

The net force on particle $A$ is $F_{B \text{ on } A}$ and the net force on particle $B$ is $F_{A \text{ on } B}$ so from Eq.(4.4) the rates of change of the momenta of the two particles are

$$
\dot{p}_A = \frac{d\dot{p}_A}{dt}, \quad \dot{p}_B = \frac{d\dot{p}_B}{dt}.
$$

(4.10)

The momentum of each particle changes, but these changes are related to each other by Newton’s third law: $F_{B \text{ on } A} = -F_{A \text{ on } B}$. Adding together the two eqs. In Eq.(4.10), we have

$$
\dot{F}_{B/A} + \dot{F}_{A/B} = \frac{d\dot{p}_A}{dt} + \frac{d\dot{p}_B}{dt} = \frac{d(\dot{p}_A + \dot{p}_B)}{dt} = 0.
$$

(4.11)
The rates of change of the two momenta are equal and opposite, so the rate of change of the vector sum $p_A + p_B$ is zero. We define the total momentum $P$ of the system of two particles as the vector sum of the momenta of the individual particles; that is,

$$\vec{p} = \vec{p}_A + \vec{p}_B.$$ \hspace{1cm} (4.12)

Then Eq. (4.11) becomes

$$\vec{F}_{B/A} + \vec{F}_{A/B} + \frac{d\vec{P}}{dt} = 0.$$ \hspace{1cm} (4.13)

The time rate of change of the total momentum $P$ is zero. Hence the total momentum of the system is constant, even though the individual momenta of the particles that make up the system can change.

If external forces are also present, they must be included on the left side of Eq. (4.13) along with the internal forces. Then the total momentum is, in general, not constant. But if the vector sum of the external forces is zero, as in Fig. 4.4, these forces have no effect on the left side of Eq. (4.13), and $dP/dt$ is again zero. Thus we have the following general result:

**If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.**

Figure 4.4 - Two ice skaters push each other as they skate on a frictionless, horizontal surface. (Compare to Fig. 4.9.)
This is the simplest form of the **principle of conservation of momentum**. This principle is a direct consequence of Newton’s third law. What makes this principle useful is that it doesn’t depend on the detailed nature of the internal forces that act between members of the system. This means that we can apply conservation of momentum even if (as is often the case) we know very little about the internal forces. We have used Newton’s second law to derive this principle, so we have to be careful to use it only in inertial frames of reference.

We can generalize this principle for a system that contains any number of particles \( A, B, C, \ldots \) interacting only with one another, with total momentum

\[
\vec{P} = \vec{P}_A + \vec{P}_B + \ldots = m_A \vec{\dot{p}}_A + m_B \vec{\dot{p}}_B + \ldots. \tag{4.14}
\]

![Diagram](image)

**Figure 4.5** - When applying conservation of momentum, remember that momentum is a vector quantity!

We make the same argument as before: The total rate of change of momentum of the system due to each action–reaction pair of internal forces is zero. Thus the total rate of change of momentum of the entire system is zero when-ever the vector sum of the external forces acting on it is zero. The internal forces can change the momenta of individual particles but not the total momentum of the system.
Conservation of momentum means conservation of its components. When you apply the conservation of momentum to a system, remember that momentum is a \textit{vector} quantity. Hence you must use vector addition to compute the total momentum of a system (\textbf{Fig. 4.5}). Using components is usually the simplest method. If \( p_{Ax} \), \( p_{Ay} \), and \( p_{Az} \) are the components of momentum of particle \( A \), and similarly for the other particles, then Eq. (4.14) is equivalent to the component equations

\[
\begin{align*}
    P_x &= p_{Ax} + p_{Bx} + \ldots, \\
    P_y &= p_{Ay} + p_{By} + \ldots, \\
    P_z &= p_{Az} + p_{Bz} + \ldots
\end{align*}
\]  

(4.15)

If the vector sum of the external forces on the system is zero, then \( P_x \), \( P_y \), and \( P_z \) are all constant.

In some ways the principle of conservation of momentum is more general than the principle of conservation of mechanical energy. For example, mechanical energy is conserved only when the internal forces are \textit{conservative}—that is, when the forces allow two-way conversion between kinetic and potential energies. But conservation of momentum is valid even when the internal forces are \textit{not} conservative. In this chapter we will analyze situations in which both momentum and mechanical energy are conserved, and others in which only momentum is conserved. These two principles play a fundamental role in all areas of physics, and we will encounter them throughout our study of physics.

\textbf{4.3 Momentum conservation and collisions}

To most people the term \textit{collision} is likely to mean some sort of automotive disaster. We’ll broaden the meaning to include any strong interaction between bodies that lasts a relatively short time. So we include not only car accidents but also balls colliding on a billiard table, neutrons hitting atomic nuclei in a nuclear reactor, and a close encounter of a spacecraft with the planet Saturn.

If the forces between the colliding bodies are much larger than any external forces, as is the case in most collisions, we can ignore the external forces and
treat the bodies as an isolated system. Then momentum is conserved and the total momentum of the system has the same value before and after the collision. Two cars colliding at an icy intersection provide a good example. Even two cars colliding on dry pavement can be treated as an isolated system during the collision if the forces between the cars are much larger than the friction forces of pavement against tires.

4.3.1 Elastic and inelastic collisions

If the forces between the bodies are also conservative, so no mechanical energy is lost or gained in the collision, the total kinetic energy of the system is the same after the collision as before. Such a collision is called an elastic collision. A collision between two marbles or two billiard balls is almost completely elastic. Figure 4.6 shows a model for an elastic collision. When the gliders collide, their springs are momentarily compressed and some of the original kinetic energy is momentarily converted to elastic potential energy. Then the gliders bounce apart, the springs expand, and this potential energy is converted back to kinetic energy.

![Figure 4.6 - Two gliders undergoing an elastic collision on a frictionless surface. Each glider has a steel spring bumper that exerts a conservative force on the other glider](image)

A collision in which the total kinetic energy after the collision is less than before the collision is called an inelastic collision. A meatball landing on a
plate of spaghetti and a bullet embedding itself in a block of wood are examples of inelastic collisions. An inelastic collision in which the colliding bodies stick together and move as one body after the collision is called a **completely inelastic collision**. **Figure 4.7** shows an example; we have replaced the spring bumpers in Fig. 4.6 with Velcro, which sticks the two bodies together.

![Figure 4.7 - Two gliders undergoing a completely inelastic collision. The spring bumpers on the gliders are replaced by Velcro, so the gliders stick together after collision](image)

Remember this rule: **In any collision in which external forces can be ignored, momentum is conserved and the total momentum before equals the total momentum after; in elastic collisions only, the total kinetic energy before equals the total kinetic energy after.**

### 4.3.2 Completely inelastic collisions

Let’s look at what happens to momentum and kinetic energy in a completely in-elastic collision of two bodies \((A \text{ and } B)\), as in Fig. 4.7. Because the two bodies stick together after the collision, they have the same final velocity \(\bar{\varphi}_2\),

\[
\bar{\varphi}_{A2} = \bar{\varphi}_{B2} = \bar{\varphi}_2,
\]

Conservation of momentum gives the relationship

\[
m_A \bar{\varphi}_{A1} + m_B \bar{\varphi}_{B1} = (m_A + m_B) \bar{\varphi}_2. \tag{4.16}
\]

If we know the masses and initial velocities, we can compute the common final velocity \(\bar{\varphi}_2\).
Suppose, for example, that a body with mass \( m_A \) and initial \( x \)-component of velocity \( \vec{\mathbf{v}}_{Ax} \) collides in elastically with a body with mass \( m_B \) that is initially at rest (\( \vec{\mathbf{v}}_{Bx1} = 0 \)). From Eq. (4.16) the common \( x \)-component of velocity \( \vec{\mathbf{v}}_{2x} \) of both bodies after the collision is

\[
\vec{\mathbf{v}}_{2x} = \frac{m_A}{m_A + m_B} \vec{\mathbf{v}}_{1x}.
\]

Let’s verify that the total kinetic energy after this completely inelastic collision is less than before the collision. The motion is purely along the \( x \)-axis, so the kinetic energies \( K_1 \) and \( K_2 \) before and after the collision, respectively, are

\[
K_1 = \frac{1}{2} m_A \vec{\mathbf{v}}_{1x}^2,
\]

\[
K_2 = \frac{1}{2} (m_A + m_B) \vec{\mathbf{v}}_{2x}^2 = \frac{1}{2} (m_A + m_B) \left( \frac{m_A}{m_A + m_B} \right)^2 \vec{\mathbf{v}}_{1x}^2.
\]

The ratio of final to initial kinetic energy is

\[
\frac{K_2}{K_1} = \frac{m_A}{m_A + m_B}.
\]

The right side is always less than unity because the denominator is always greater than the numerator. Even when the initial velocity of \( m_B \) is not zero, the kinetic energy after a completely inelastic collision is always less than before.

Please note: Don’t memorize Eq. (4.17) or (4.18) We derived them only to prove that kinetic energy is always lost in a completely inelastic collision.

### 4.4 Elastic collisions

We saw that an elastic collision in an isolated system is one in which kinetic energy (as well as momentum) is conserved. Elastic collisions occur when the forces between the colliding bodies are conservative. When two billiard balls collide, they squash a little near the surface of contact, but then
they spring back. Some of the kinetic energy is stored temporarily as elastic potential energy, but at the end it is reconverted to kinetic energy (Fig. 4.8).

Figure 4.8 - Billiard balls deform very little when they collide, and they quickly spring back from any deformation they do undergo. Hence the force of interaction between the balls is almost perfectly conservative, and the collision is almost perfectly elastic.

Let’s look at a one-dimensional elastic collision between two bodies A and B, in which all the velocities lie along the same line. We call this line the x-axis, so each momentum and velocity has only an x-component. We call the x-velocities before the collision \( \mathcal{A}_{Ax} \) and \( \mathcal{B}_{Bx} \), and those after the collision \( \mathcal{A}_{A2x} \) and \( \mathcal{B}_{B2x} \). From conservation of kinetic energy we have

\[
\frac{1}{2} m_A \mathcal{A}_{Ax}^2 + \frac{1}{2} m_B \mathcal{B}_{Bx}^2 = \frac{1}{2} m_A \mathcal{A}_{A2x}^2 + \frac{1}{2} m_B \mathcal{B}_{B2x}^2.
\]

and conservation of momentum gives

\[
A \mathcal{A}_{Ax} + m_B \mathcal{B}_{Bx} = m_A \mathcal{A}_{A2x} + m_B \mathcal{B}_{B2x}.
\]

If the masses \( m_A \) and \( m_B \) and the initial velocities \( \mathcal{A}_{Ax} \) and \( \mathcal{B}_{Bx} \) are known, we can solve these two equations to find the two final velocities \( \mathcal{A}_{A2x} \) and \( \mathcal{B}_{B2x} \).

**4.4.1 Elastic collisions, one body initially at rest**

The general solution to the above equations is a little complicated, so we will concentrate on the particular case in which body B is at rest before the collision (so \( \mathcal{B}_{Bx} = 0 \)). Think of body B as a target for body A to hit. Then the kinetic energy and momentum conservation equations are, respectively,
\[
\frac{1}{2} m_A g_{A1x}^2 = \frac{1}{2} m_A g_{A2x}^2 + \frac{1}{2} m_B g_{B2x}^2, \tag{4.19}
\]

\[
m_A g_{A1x} = m_A g_{A2x} + m_B g_{B2x}. \tag{4.20}
\]

We can solve for \( g_{A2x} \) and \( g_{B2x} \) in terms of the masses and the initial velocity \( g_{A1x} \). This involves some fairly strenuous algebra, but it’s worth it. No pain, no gain! The simplest approach is somewhat indirect, but along the way it uncovers an additional interesting feature of elastic collisions.

First we rearrange Eqs. (4.19) and (4.20) as follows:

\[
m_B g_{B2x}^2 = m_A \left( g_{A1x}^2 - g_{A2x}^2 \right) = m_A \left( g_{A1x} - g_{A2x} \right) \left( g_{A1x} + g_{A2x} \right). \tag{4.21}
\]

\[
m_B g_{B2x} = m_A \left( g_{A1x} - g_{A2x} \right). \tag{4.22}
\]

Now we divide Eq. (4.21) by Eq. (4.22) to obtain

\[
g_{B2x} = g_{A1x} + g_{A2x}. \tag{4.23}
\]

We substitute this expression back into Eq. (4.22) to eliminate \( g_{B2x} \) and then solve for \( g_{A2x} \):

\[
m_B \left( g_{A1x} - g_{A2x} \right) = m_A \left( g_{A1x} - g_{A2x} \right),
\]

\[
g_{A2x} = \frac{m_A - m_B}{m_A + m_B} g_{A1x}. \tag{4.24}
\]

Finally, we substitute this result back into Eq. (4.23) to obtain

\[
g_{B2x} = \frac{2m_A}{m_A + m_B} g_{A1x}. \tag{4.25}
\]
Now we can interpret the results. Suppose $A$ is a Ping-Pong ball and $B$ is a bowling ball. Then we expect $A$ to bounce off after the collision with a velocity nearly equal to its original value but in the opposite direction (Fig. 4.9a), and we expect $B$’s velocity to be much less. That’s just what the equations predict. When $m_A$ is much smaller than $m_B$, the fraction in Eq. (4.24) is approximately
equal to -1, so $\mathcal{J}_{A2x}$ is approximately equal to $-\mathcal{J}_{A1x}$. The fraction in Eq. (4.25) is much smaller than unity, so $\mathcal{J}_{B2x}$ is much less than $\mathcal{J}_{A1x}$. Figure 4.9b shows the opposite case, in which $A$ is the bowling ball and $B$ the Ping-Pong ball and $m_A$ is much larger than $m_B$. What do you expect to happen then? Check your predictions against Eqs. (4.24) and (4.25). Another interesting case occurs when the masses are equal (Fig. 4.10). If $m_A = m_B$, then Eqs. (4.24) and (4.25) give $\mathcal{J}_{A2x} = 0$ and $\mathcal{J}_{B2x} = \mathcal{J}_{A1x}$. That is, the body that was moving stops dead; it gives all its momentum and kinetic energy to the body that was at rest. This behavior is familiar to all pool players.

4.4.2 Elastic collisions and relative velocity

Let’s return to the more general case in which $A$ and $B$ have different masses. Equation (4.23) can be rewritten as

$$
\mathcal{J}_{A1x} = \mathcal{J}_{B2x} - \mathcal{J}_{A2x}
$$

(4.26)

Here $\mathcal{J}_{B2x} - \mathcal{J}_{A2x}$ is the velocity of $B$ relative to $A$ after the collision; from Eq. (4.26), this equals $\mathcal{J}_{A1x}$, which is the negative of the velocity of $B$ relative to $A$ before the collision. The relative velocity has the same magnitude, but opposite sign, before and after the collision. The sign changes because $A$ and $B$ are approaching each other before the collision but moving apart after the collision. If we view this collision from a second coordinate system moving with constant velocity relative to the first, the velocities of the bodies are different but the relative velocities are the same. Hence our statement about relative velocities holds for any straight-line elastic collision, even when neither body is at rest initially. In a straight-line elastic collision of two bodies, the relative velocities before and after the collision have the same magnitude but opposite sign. This means that if $B$ is moving before the collision, Eq. (4.26) becomes

$$
\mathcal{J}_{B2x} - \mathcal{J}_{A2x} = - (\mathcal{J}_{B1x} - \mathcal{J}_{A1x})
$$

(4.27)
It turns out that a vector relationship similar to Eq. (4.27) is a general property of all elastic collisions, even when both bodies are moving initially and the velocities do not all lie along the same line. This result provides an alternative and equivalent definition of an elastic collision: In an elastic collision, the relative velocity of the two bodies has the same magnitude before and after the collision. Whenever this condition is satisfied, the total kinetic energy is also conserved.

When an elastic two-body collision isn’t head-on, the velocities don’t all lie along a single line. If they all lie in a plane, then each final velocity has two unknown components, and there are four unknowns in all. Conservation of energy and conservation of the \(x\)- and \(y\)-components of momentum give only three equations. To determine the final velocities uniquely, we need additional information, such as the direction or magnitude of one of the final velocities.

4.5. Center of mass

We can restate the principle of conservation of momentum in a useful way by using the concept of center of mass. Suppose we have several particles with masses \(m_1, m_2, \) and so on. Let the coordinates of \(m_1\) be \((x_1, y_1)\), those of \(m_2\) be \((x_2, y_2)\), and so on. We define the center of mass of the system as the point that has coordinates \(x_{cm}, y_{cm}\) given by

\[
x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i x_i}{\sum_i m_i}.
\]

\[
y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i y_i}{\sum_i m_i}.
\]

We can express the position of the center of mass as a vector \(\mathbf{r}_{cm}\):
We say that the center of mass is a \textit{mass-weighted average} position of the particles.

\subsection{4.5.1. Motion of the center of mass}

For solid bodies, in which we have (at least on a macroscopic level) a continuous distribution of matter, the sums in Eqs. (4.28) have to be replaced by integrals. The calculations can get quite involved, but we can say three general things about such problems (Fig. 4.11). First, whenever a homogeneous body has a geometric center, such as a billiard ball, a sugar cube, or a can of frozen orange juice, the center of mass is at the geometric center. Second, whenever a body has an axis of symmetry, such as a wheel or a pulley, the center of mass always lies on that axis. Third, there is no law that says the center of mass has to be within the body. For example, the center of mass of a donut is in the middle of the hole.

\begin{equation}
\vec{r}_{\text{cm}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}.
\end{equation}

Figure 4.11 - Locating the center of mass of a symmetric object
derivatives of $x_{cm}$ and $y_{cm}$. Also, $dx_1/dt$ is the $x$-component of velocity of particle 1, so $dx_1/dt = \dot{x}_1$, and so on. Taking time derivatives of Eqs. (4.28), we get

$$
\dot{\mathbf{\mathcal{G}}}_{cm} = \frac{m_1 \mathcal{G}_{1x} + m_2 \mathcal{G}_{2x} + m_3 \mathcal{G}_{3x} + \ldots}{m_1 + m_2 + m_3 + \ldots},
$$

$$
\dot{\mathbf{\mathcal{G}}}_{cm-y} = \frac{m_1 \mathcal{G}_{1y} + m_2 \mathcal{G}_{2y} + m_3 \mathcal{G}_{3y} + \ldots}{m_1 + m_2 + m_3 + \ldots}.
$$

These equations are equivalent to the single vector equation obtained by taking the time derivative of Eq. (4.29):

$$
\ddot{\mathbf{\mathcal{G}}}_{cm} = \frac{m_1 \ddot{\mathcal{G}}_{1x} + m_2 \ddot{\mathcal{G}}_{2x} + m_3 \ddot{\mathcal{G}}_{3x} + \ldots}{m_1 + m_2 + m_3 + \ldots}, \tag{4.30}
$$

We denote the total mass $m_1 + m_2 + g$ by $M$. We can then rewrite Eq. (4.30) as

$$
M \ddot{\mathbf{\mathcal{G}}}_{cm} = m_1 \ddot{\mathcal{G}}_{1} + m_2 \ddot{\mathcal{G}}_{2} + m_3 \ddot{\mathcal{G}}_{3} + \ldots = \ddot{\mathbf{P}}. \tag{4.31}
$$

So the total momentum $\mathbf{P}$ of a system equals the total mass times the velocity of the center of mass. When you catch a baseball, you are really catching a collection of a very large number of molecules of masses $m_1, m_2, m_3, \ldots$. The impulse you feel is due to the total momentum of this entire collection. But this impulse is the same as if you were catching a single particle of mass $M = m_1 + m_2 + m_3 + g$ moving with $\ddot{\mathbf{\mathcal{G}}}_{sm}$, the velocity of the collection’s center of mass. So Eq. (4.31) helps us justify representing an extended body as a particle.

For a system of particles on which the net external force is zero, so that the total momentum $\mathbf{P}$ is constant, the velocity of the center of mass $\ddot{\mathbf{\mathcal{G}}}_{sm} = \mathbf{P}/M$ is also constant. Figure 4.11 shows an example. The overall motion of the wrench appears complicated, but the center of mass follows a straight line, as though all the mass were concentrated at that point.
5 Dynamics of Rotational Motion

5.1 Torque

In this chapter we’ll define a new physical quantity, torque, that describes the twisting or turning effort of a force. We’ll find that the net torque acting on a rigid body determines its angular acceleration, in the same way that the net force on a body determines its linear acceleration. We’ll also look at work and power in rotational motion so as to understand, for example, how energy is transferred by an electric motor. Next we’ll develop a new conservation principle, conservation of angular momentum, that is tremendously useful for understanding the rotational motion of both rigid and no rigid bodies. We’ll finish this chapter by studying gyroscopes, rotating devices that don’t fall over when you might think they should—but that actually behave in accordance with the dynamics of rotational motion.

![Diagram of forces on a wrench]

Figure 10.1 - Which of these three equal-magnitude forces is most likely to loosen the tight bolt?

We know that forces acting on a body can affect its translational motion—that is, the motion of the body as a whole through space. Now we want to learn which aspects of a force determine how effective it is in causing or changing rotational motion. The magnitude and direction of the force are important, but so is the point on the body where the force is applied. In Fig. 5.1 a wrench is being used to loosen a tight bolt. Force \( \vec{F}_a \), applied near the end of the handle, is more effective than an equal force \( \vec{F}_a \) applied near the bolt. Force
\( \vec{F}_c \) does no good; it’s applied at the same point and has the same magnitude as \( \vec{F}_b \), but it’s directed along the length of the handle. The quantitative measure of the tendency of a force to cause or change a body’s rotational motion is called torque; we say that \( \vec{F}_a \) applies a torque about point \( O \) to the wrench in Fig. 5.1, \( \vec{F}_b \) applies a greater torque about \( O \), and \( \vec{F}_c \) applies zero torque about \( O \).

Figure 5.2 shows three examples of how to calculate torque. The body can rotate about an axis that is perpendicular to the plane of the figure and passes through point \( O \). Three forces act on the body in the plane of the figure. The tendency of the first of these forces, \( \vec{F}_1 \), to cause a rotation about \( O \) depends on its magnitude \( \vec{F}_1 \). It also depends on the perpendicular distance \( l_1 \) between point \( O \) and the line of action of the force (that is, the line along which the force vector lies). We call the distance \( l_1 \) the lever arm (or moment arm) of force \( \vec{F}_1 \) about \( O \). The twisting effort is directly proportional to both \( \vec{F}_1 \) and \( l_1 \), so we define the torque (or moment) of the force \( \vec{F}_1 \) with respect to \( O \) as the product \( \vec{F}_1 l_1 \). We use the Greek letter \( \tau \) (tau) for torque. If a force of magnitude \( F \) has a line of action that is a perpendicular distance \( l \) from \( O \), the torque is

\[
\tau = F \cdot l.
\]  

(5.1)

Physicists usually use the term “torque,” while engineers usually use “moment” (unless they are talking about a rotating shaft). The lever arm of \( \vec{F}_1 \) in
Fig. 5.2 is the perpendicular distance $l_1$, and the lever arm of $\vec{F}_2$ is the perpendicular distance $l_2$. The line of action of $\vec{F}_3$ passes through point $O$, so the lever arm for $\vec{F}_3$ is zero and its torque with respect to $O$ is zero. In the same way, force $\vec{F}_c$ in Fig. 5.1 has zero torque with respect to point $O$; $\vec{F}_b$ has a greater torque than $\vec{F}_a$ because its lever arm is greater.

Force $\vec{F}_1$ in Fig. 5.2 tends to cause *counterclockwise* rotation about $O$, while $\vec{F}_2$ tends to cause *clockwise* rotation. To distinguish between these two possibilities, we need to choose a positive sense of rotation. With the choice that *counterclockwise* $\vec{F}_1$ and $\vec{F}_2$ about $O$ are

$$\tau_1 = +F_1 \cdot l_1, \tau_2 = -F_2 \cdot l_2.$$

Figure 5.2 shows this choice for the sign of torque. We will often use the symbol $+$ to indicate our choice of the positive sense of rotation.

The SI unit of torque is the newton-meter. In our discussion of work and energy we called this combination the joule. But torque is *not* work or energy, and torque should be expressed in newton-meters, *not* joules.

**5.2 Torque and Angular Acceleration for a Rigid Body**

We’re now ready to develop the fundamental relationship for the rotational dynamics of a rigid body. We’ll show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

We choose the axis of rotation to be the $z$-axis; the first particle has mass $m_1$ and distance $r_1$ from this axis (Fig. 5.3). The *net force* $\vec{F}_1$ acting on this particle has a component $F_{1,\text{rad}}$ along the radial direction, a component $\vec{F}_{1,\text{tan}}$ that is tangent to the circle of radius $r_1$ in which the particle moves as the body rotates, and component $\vec{F}_{1,z}$ along the axis of rotation. Newton’s second law for the tangential component is
\[ F_{1,z} r_1 = m_1 a_{1,z}, \]  
(5.3)  
\[ F_{1,z} r_1 = m_1 r_1^2 \alpha_z. \]  
(5.4)  

The subscript \( z \) is a reminder that the torque affects rotation around the \( z \) -axis, in the same way that the subscript on \( \vec{F}_{1,z} \) is a reminder that this force affects the motion of particle 1 along the \( z \)-axis.

Neither of the components \( \vec{F}_1, \text{rad} \) or \( \vec{F}_{1,z} \) contributes to the torque about the \( z \) -axis, since neither tends to change the particle’s rotation about that axis. So \( \tau_z = \vec{F}_{1,\text{tan}} r_1 \), \( z \) is the total torque acting on the particle with respect to the rotation axis. Also, \( m_i r_i^2 \) is \( I_1 \), the moment of inertia of the particle about the rotation axis. Hence we can rewrite Eq. (5.4) as

\[ \tau_z = I_1 \alpha_z = m_1 r_1^2 \alpha_z. \]

We write such an equation for every particle in the body, then add all these equations:

\[ \tau_{1z} + \tau_{2z} + \ldots = I_1 \alpha_z + I_2 \alpha_z + \ldots = m_1 r_1^2 \alpha_z + m_1 r_2^2 \alpha_z + \ldots \]

\[ \sum \tau_{iz} = (\sum m_i r_i^2) \alpha_z. \]  
(5.5)  

The left side of Eq. (5.5) is the sum of all the torques about the rotation axis that act on all the particles. The right side is \( I = \sum m_i r_i^2 \), the total moment of inertia about the rotation axis, multiplied by the angular acceleration \( \alpha_z \) Note
that $a_z$ is the same for every particle because this is a rigid body. Thus Eq. (5.5) says that for the rigid body as a whole,

$$\sum\tau_z = Ia_z, \quad (5.6)$$

Where $\tau$ is the net torque on a rigid body about axis; $I$ is the moment of inertia; $a$ is the angular acceleration.

### 5.3.1 Combined Translation and Rotation: Energy Relationships

It’s beyond our scope to prove that rigid-body motion can always be divided into translation of the center of mass and rotation about the center of mass. But we can prove this for the kinetic energy $K$ of a rigid body that has both translational and rotational motions. For such a body, $K$ is the sum of two parts:

$$K = \frac{1}{2}M\vec{\omega}_{cm}^2 + \frac{1}{2}I_{cm}\omega^2. \quad (5.7)$$

To prove this relationship, we again imagine the rigid body to be made up of particles. For a typical particle with mass $m_i$ (Fig. 5.4), the velocity $\vec{\dot{r}}_i$ of this particle relative to an inertial frame is the vector sum of the velocity $\vec{\dot{r}}_{cm}$ of the center of mass and the velocity $\vec{\dot{r}}'_i$ of the particle relative to the center of mass:

$$\vec{\dot{r}}_i = \vec{\dot{r}}_{cm} + \vec{\dot{r}}'_i. \quad (5.8)$$

The kinetic energy $K_i$ of this particle in the inertial frame is $\frac{1}{2}m_i\vec{\dot{r}}_i^2$, which we can also express as $\frac{1}{2}m_i(\vec{\dot{r}} \cdot \vec{\dot{r}}_i)$. Substituting Eq. (5.8) into this, we get

$$K_i = \frac{1}{2}m_i(\vec{\dot{r}}_{cm} + \vec{\dot{r}}'_i) \cdot (\vec{\dot{r}}_{cm} + \vec{\dot{r}}'_i),$$

$$= \frac{1}{2}m_i(\vec{\dot{r}}_{cm} \cdot \vec{\dot{r}}_{cm} + 2\vec{\dot{r}}_{cm} \cdot \vec{\dot{r}}'_i + \vec{\dot{r}}'_i \cdot \vec{\dot{r}}).$$

The total kinetic energy is the sum $K_i$ for all the particles making up the body. Expressing the three terms in this equation as separate sums, we get
\[
K = \sum K_i = \sum \left( \frac{1}{2} m_i \mathbf{\dot{r}}_i \cdot \mathbf{\dot{r}}_i \right) + \sum (m_i \mathbf{\dot{r}}_i \cdot \mathbf{\dot{r}}_j) + \sum \left( \frac{1}{2} m_i \mathbf{\dot{r}}_i^2 \right).
\]

The first and second terms have common factors that we take outside the sum:

\[
K = \frac{1}{2} \left( \sum m_i \mathbf{\dot{r}}_i \right)^2 + \sum m_i \mathbf{\dot{r}}_i \cdot \sum m_j \mathbf{\dot{r}}_j + \sum \left( m_i \mathbf{\dot{r}}_i^2 \right).
\]  

(5.9)

The second term is zero because \( \sum m_i \mathbf{\dot{r}}_i \) is \( M \) times the velocity of the center of mass relative to the center of mass, and this is zero by definition. The last term is the sum of the kinetic energies of the particles computed by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation around the center of mass. (5.10) becomes Eq. (5.7):

\[
K = \frac{1}{2} M \mathbf{\bar{v}}_c^2 + \frac{1}{2} I_{cm} \omega^2.
\]  

(5.10)

5.3.2 Rolling Without Slipping

An important case of combined translation and rotation is **rolling without slipping**. The rolling wheel in Fig. 5.4 is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest. In this frame, the point on the wheel that contacts the surface must be instantaneously *at rest* so that it does not slip. Hence the velocity \( \mathbf{\dot{r}}_i \) of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity \( \mathbf{\dot{r}}_{cm} \). If the wheel’s radius is \( R \) and its angular speed about the center of mass is \( \omega \) then the magnitude of \( \mathbf{\dot{r}}_i \) is \( Rw \); hence

\[
\mathbf{\dot{r}}_{cm} = R \omega,
\]  

(5.11)

Where \( \mathbf{\dot{r}}_{cm} \) is the speed of center of mass; \( R \) is the radius of the wheel; \( \omega \) is the Angular speed of a wheel.
Figure 5.4 - The motion of a rolling wheel is the sum of the translational motion of the center of mass plus the rotational motion of the wheel around the center of mass.

As Fig. 5.4 shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward twice as fast as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

At any instant we can think of the wheel as rotating about an “instantaneous axis” of rotation that passes through the point of contact with the ground. The angular velocity \( v \) is the same for this axis as for an axis through the center of mass; an observer at the center of mass sees the rim make the same number of revolutions per second as does an observer at the rim watching the center of mass spin around him. If we think of the motion of the rolling wheel in Fig. 5.4 in this way, the kinetic energy of the wheel is

\[
K = \frac{1}{2} I_1 \omega^2,
\]

where \( I_1 \) is the moment of inertia of the wheel about an axis through point 1. But by the parallel-axis theorem, \( I_1 = I_{cm} + MR^2 \), where \( M \) is the total mass of the wheel and \( I_{cm} \) is the moment of inertia with respect to an axis through the center of mass. Using Eq. (5.11), we find that the wheel’s kinetic energy is as given by Eq. (5.7):

\[
K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} MR^2 \omega^2 = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M \theta_{cm}^2.
\]

(5.12)
5.4 Work and Power in Rotational Motion

When you pedal a bicycle, you apply forces to a rotating body and do work on it. Similar things happen in many other real-life situations, such as a rotating motor shaft driving a power tool or a car engine propelling the vehicle. Let’s see how to apply our ideas about work from Chapter 3 to rotational motion.

Suppose a tangential force $\vec{F}_{\text{tan}}$ acts at the rim of a pivoted disk—for example, a child running while pushing on a playground merry-go-round (Fig. 5.5a).

The disk rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval $dt$ (Fig. 5.5b). The work $dW$ done by the force $\vec{F}_{\text{tan}}$ while a point on the rim moves a distance $ds$ is $dA=\vec{F}_{\text{tan}} \cdot ds$. If $d\theta$ is measured in radians, then $dS=Rd\theta$ and $dA=\vec{F}_{\text{tan}} \cdot R \, d\theta$.

Now $\vec{F}_{\text{tan}} \cdot R$ is the torque $\tau_z$ due to the force $\vec{F}_{\text{tan}}$, so

$$dA=\tau_z \, d\theta. \quad (5.13)$$

As the disk rotates from $\theta_1$ to $\theta_2$, the total work done by the torque is

$$A=\int_{\theta_1}^{\theta_2} \tau_z \, d\theta. \quad (5.14)$$

If the torque remains constant while the angle changes, then the work is the product of torque and angular displacement:

$$A=\tau_z (\theta_2 - \theta_1) = \tau_z \Delta \theta. \quad (5.15)$$

If torque is expressed in newton-meters N m and angular displacement in radians, the work is in joules.
Figure 5.5 - A tangential force applied to a rotating body does work

If the force in Fig. 5.5 had an axial component (parallel to the rotation axis) or a radial component (directed toward or away from the axis), that component would do no work because the displacement of the point of application has only a tangential component. An axial or radial component of force would also make no contribution to the torque about the axis of rotation. So Eqs. (5.14) and (5.15) are correct for any force, no matter what its components.

We then transform the integrand in Eq. (5.14) into an integrand with respect to $\omega$ as follows:

$$\tau_z = d\theta = (I_\alpha) d\theta = I \frac{d\alpha}{dt} d\theta = I \frac{d\theta}{dt} d\omega_z = I \omega_z d\omega_z.$$  

Since $\tau_z$ is the net torque, the integral in Eq. (5.14) is the total work done on the rotating rigid body. This equation then becomes

$$W_{\text{rot}} = \int_{\alpha_k}^{\alpha_f} I \omega_z d\omega_z = \frac{1}{2} I \omega_z^2 - \frac{1}{2} \omega_0^2.$$  

(5.16)

The change in the rotational kinetic energy of a rigid body equals the work done by forces exerted from outside the body (Fig. 5.6).

How does power relate to torque? When we divide both sides of Eq. (5.13) by the time interval $dt$ during which the angular displacement occurs, we find

$$\frac{dA}{dt} = \tau_z \frac{d\theta}{dt}.$$  


But \( dA/dt \) is the rate of doing work, or power \( P \), and \( d\theta/dt \) is angular velocity \( \omega \):

\[
P = \tau \cdot \omega. \tag{5.17}
\]

Figure 5.6 - The rotational kinetic energy of a helicopter’s main rotor is equal to the total work done to set it spinning. When it is spinning at a constant rate, positive work is done on the rotor by the engine and negative work is done on it by air resistance. Hence the net work being done is zero and the kinetic energy remains constant.

5.5 Angular Momentum

Its relationship to momentum \( \vec{p} \) is exactly the same as the relationship of torque to force, \( \tau = \vec{r} \times \vec{F} \). For a particle with constant mass \( m \) and velocity \( \vec{\theta} \), the angular momentum is

\[
\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{\theta}. \tag{5.18}
\]

Figure 5.7 - Calculating the angular momentum \( \vec{L} = \vec{r} \times m\vec{\theta} = \vec{r} \times \vec{p} \) of a particle with mass \( m \) moving in the \( xy \)-plane.
The value of \( \vec{L} \) depends on the choice of origin \( O \), since it involves the particle’s position vector \( \vec{r} \) relative to \( O \). The units of angular momentum are \( \text{kg} \cdot \text{m}^2/\text{s} \). In Fig. 5.7 a particle moves in the \( xy \)-plane; its position vector \( \vec{r} \) and momentum \( \vec{p} = m \vec{\omega} \) are shown. The angular momentum vector \( \vec{L} \) is perpendicular to the \( xy \)-plane. The right-hand rule for vector products shows that its direction is along the \(+z\)-axis, and its magnitude is

\[
L = m \vec{\omega} \sin \varphi = m \vec{\omega} l,
\]

where \( l \) is the perpendicular distance from the line of \( \vec{\omega} \) to \( O \). This distance plays the role of “lever arm” for the momentum vector. When a net force \( \vec{F} \) acts on a particle, its velocity and momentum change, so its angular momentum may also change. We can show that the rate of change of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (5.18), using the rule for the derivative of a product:

\[
\frac{d\vec{L}}{dt} = (\frac{d\vec{r}}{dt} \times m \vec{\omega}) + (\vec{r} \times m \frac{d\vec{\omega}}{dt}) = (\vec{\omega} \times m \vec{\omega}) + (\vec{r} \times m \vec{a}).
\]

The first term is zero because it contains the vector product of the vector \( \vec{\omega} = \frac{d\vec{r}}{dt} \) with itself. In the second term we replace \( m \vec{a} \) with the net force \( \vec{F} \):

\[
\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}.
\]

The rate of change of angular momentum of a particle equals the torque of the net force acting on it.

Figure 5.8 - Calculating the angular momentum of a particle of mass \( m_i \) in a rigid body rotating at angular speed \( \vec{\omega} \).
We can use Eq. (5.19) to find the total angular momentum of a rigid body rotating about the z-axis with angular speed \( \omega \). First consider a thin slice of the body lying in the xy-plane (Fig. 5.8). Each particle in the slice moves in a circle centered at the origin, and at each instant its velocity \( \vec{\beta}_i \) is perpendicular to its position vector \( \vec{r}_i \), as shown. Hence in Eq. (5.19), \( \varphi=90^\circ \) for every particle. A particle with mass \( m_i \) at a distance \( r_i \) from \( O \) has a speed \( \vec{\beta}_i = \omega r_i \). From Eq. (5.19) the magnitude \( L_i \) of its angular momentum is

\[
L_i = m_i (r_i \omega) = m_i r_i^2 \omega. \tag{5.21}
\]

The direction of each particle’s angular momentum, as given by the right-hand rule for the vector product, is along the \( +z \)-axis.

The total angular momentum of the slice of the body lying in the xy-plane is the sum \( \sum L_i \) of the angular momenta \( L_i \) of the particles. Summing Eq. (5.21), we have

\[
L = \sum L_i = (\sum m_i r_i^2) \omega = I \omega,
\]

where \( I \) is the moment of inertia of the slice about the z-axis.

We can do this same calculation for the other slices of the body, all parallel to the xy-plane. For points that do not lie in the xy-plane, a complication arises because the \( r \) vectors have components in the \( z \)-direction as well as in the \( x \)- and \( y \)-directions.
y-directions; this gives the angular momentum of each particle a component perpendicular to the z-axis. But if the z-axis is an axis of symmetry, the perpendicular components for particles on opposite sides of this axis add up to zero (Fig. 5.8). So when a body rotates about an axis of symmetry, its angular momentum vector $L$ lies along the symmetry axis, and its magnitude is $L=I\omega$.

Hence for a rigid body rotating around an axis of symmetry, $\vec{L}$ and $\vec{\omega}$ are in the same direction (Fig. 5.10). So we have the vector relationship

$$\vec{L} = I\vec{\omega}. \quad (5.22)$$

Figure 5.10 - For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis. The directions of both vectors are given by the right-hand rule.

From Eq. (5.20) the rate of change of angular momentum of a particle equals the torque of the net force acting on the particle. For any system of particles (including both rigid and no rigid bodies), the rate of change of the total angular momentum equals the sum of the torques of all forces acting on all the particles. The torques of the internal forces add to zero if these forces act along the line from one particle to another, and so the sum of the torques includes only the torques of the external forces.

$$\sum \tau_z = \frac{dL}{dt}. \quad (5.23)$$

### 5.6 Conservation of Angular Momentum

It also forms the basis for the principle of conservation of angular momentum. Like conservation of energy and of linear momentum, this principle is a universal conservation law, valid at all scales from atomic and nuclear
systems to the motions of galaxies. This principle follows directly from Eq. (5.23):
\[ \sum \vec{\tau} = d\vec{L}/dt. \]
If \( \sum \vec{\tau} = 0 \) then \( d\vec{L}/dt = 0 \) and \( \vec{L} \) is constant.

A circus acrobat, a diver, and an ice skater pirouetting on one skate all take advantage of this principle. Suppose an acrobat has just left a swing; she has her arms and legs extended and is rotating counterclockwise about her center of mass. When she pulls her arms and legs in, her moment of inertia \( I_{cm} \) with respect to her center of mass changes from a large value \( I_1 \) to a much smaller value \( I_2 \). The only external force acting on her is her weight, which has no torque with respect to an axis through her center of mass. So her angular momentum \( \vec{L} = I_{cm} \vec{\omega} \) remains constant, and her angular velocity \( \vec{\omega} \) increases as \( I_{cm} \) decreases. That is,
\[ I_1 \vec{\omega}_1 = I_2 \vec{\omega}_2. \]

When a skater or ballerina spins with arms outstretched and then pulls her arms in, her angular velocity increases as her moment of inertia decreases. In each case there is conservation of angular momentum in a system in which the net external torque is zero.

When a system has several parts, the internal forces that the parts exert on one another cause changes in the angular momenta of the parts, but the total angular momentum doesn’t change. Here’s an example. Consider two bodies A and B that interact with each other but not with anything else, (such as the astronauts we discussed in Section 4) (Fig. 5.11). Suppose body A exerts a force \( \vec{F}_{A/B} \) on body B; the corresponding torque (with respect to whatever point we choose) is \( \vec{\tau}_{A/B} \). According to Eq. (5.23), this torque is equal to the rate of change of angular momentum of B:
\[ \vec{\tau}_{A/B} = \frac{d\vec{L}_B}{dt}, \]
\[ \vec{\tau}_{B/A} = \frac{d\vec{L}_A}{dt}. \]

Figure 5.11 - Two ice skaters push each other as they skate on a frictionless, horizontal surface.

From Newton’s third law, \( \vec{F}_{B/A} = -\vec{F}_{A/B} \). Thus the torques of these two forces are equal and opposite, and \( \vec{\tau}_{A/B} = \vec{\tau}_{B/A} \). So if we add the two preceding equations, we find

\[ \frac{d\vec{L}_A}{dt} + \frac{d\vec{L}_B}{dt} = 0, \]

or, because \( \vec{L}_A + \vec{L}_B \) is the total angular momentum \( \vec{L} \) of the system,

\[ \frac{d\vec{L}}{dt} = 0. \quad (5.25) \]

That is, the total angular momentum of the system is constant. The torques of the internal forces can transfer angular momentum from one body to the other, but they can’t change the total angular momentum of the system.
6 Gravitation

6.1 Newton’s Law of Gravitation

Some of the earliest investigations in physical science started with questions that people asked about the night sky. Why doesn’t the moon fall to earth? Why do the planets move across the sky? Why doesn’t the earth fly off into space rather than remaining in orbit around the sun? The study of gravitation provides the answers to these and many related questions.

As we remarked in Chapter 2, gravitation is one of the four classes of interactions found in nature, and it was the earliest of the four to be studied extensively. Newton discovered in the 17th century that the same interaction that makes an apple fall out of a tree also keeps the planets in their orbits around the sun. This was the beginning of celestial mechanics, the study of the dynamics of objects in space. Today, our knowledge of celestial mechanics allows us to determine how to put a satellite into any desired orbit around the earth or to choose just the right trajectory to send a spacecraft to another planet.

In this chapter you will learn the basic law that governs gravitational interactions. This law is universal: Gravity acts in the same fundamental way between the earth and your body, between the sun and a planet, and between a planet and one of its moons. We’ll apply the law of gravitation to phenomena such as the variation of weight with altitude, the orbits of satellites around the earth, and the orbits of planets around the sun.

The gravitational attraction that’s most familiar to you is your weight, the force that attracts you toward the earth. By studying the motions of the moon and planets, Newton discovered a fundamental law of gravitation that describes the gravitational attraction between any two bodies. Newton published this law in 1687 along with his three laws of motion. In modern language, it says

Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the
particles and inversely proportional to the square of the distance between them.

Figure 6.1 depicts this law, which we can express as an equation:

\[ F_g = \frac{G m_1 m_2}{r^2} , \tag{6.1} \]

Where \( F \) is the Newton’s law of gravitation: The magnitude of attractive gravitational force between any two particles, \( G \) is the gravitational constant, \( m_1, m_2 \) are masses of particles, \( r \) is the distance between particles.

The gravitational constant \( G \) in Eq. (6.1) is a fundamental physical constant that has the same value for any two particles. We’ll see shortly what the value of \( G \) is and how this value is measured.

Equation (6.1) tells us that the gravitational force between two particles decreases with increasing distance \( r \): If the distance is doubled, the force is only one-fourth as great, and so on. Although many of the stars in the night sky are far more massive than the sun, they are so far away that their gravitational force on the earth is negligibly small.

We have stated the law of gravitation in terms of the interaction between two particles. It turns out that the gravitational interaction of any two bodies having spherically symmetric mass distributions (such as solid spheres or spherical shells) is the same as though we concentrated all the mass of each at its center, as in Fig. 6.2. Thus, if we model the earth as a spherically symmetric body with mass \( m_E \), the force it exerts on a particle or on a spherically symmetric body with mass \( m \), at a distance \( r \) between centers, is

\[ F_g = \frac{G m_Em}{r^2} . \tag{6.2} \]
provided that the body lies outside the earth. A force of the same magnitude is exerted on the earth by the body.

(a) The gravitational force between two spherically symmetric masses \( m_1 \) and \( m_2 \)

(b) ... is the same as if we concentrated all the mass of each sphere at the sphere’s center.

Figure 6.2 - The gravitational effect outside any spherically symmetric mass distribution is the same as though all of the mass were concentrated at its center

6.2 Weight

We defined the weight of a body in Section 1 as the attractive gravitational force exerted on it by the earth. We can now broaden our definition and say that the weight of a body is the total gravitational force exerted on the body by all other bodies in the universe. When the body is near the surface of the earth, we can ignore all other gravitational forces and consider the weight as just the earth’s gravitational attraction. At the surface of the moon we consider a body’s weight to be the gravitational attraction of the moon, and so on.

If we again model the earth as a spherically symmetric body with radius \( R_E \), the weight of a small body at the earth’s surface (a distance \( R_E \) from its center) is

\[
P = F_g = \frac{G m_E m}{R_E^2},
\]

(6.3)

Where \( P \) is the weight of a body at the earth’s surface, \( F_g \) is the gravitational force the earth exerts on body, \( G \) is the gravitational constant, \( m_E \) is the mass of the earth, \( m \) is the mass of body, \( R_E \) is the radius of the earth.
But we also know from Section 4.4 that the weight $P$ of a body is the force that causes the acceleration $g$ of free fall, so by Newton’s second law, $P=mg$. Equating this with Eq. (13.3) and dividing by $m$, we find

$$g = \frac{Gm_E}{R_E^2},$$  

(6.4)

Where $g$ is the acceleration due to gravity at the earth’s surface, $G$ is the gravitational constant, $m_E$ is the mass of the earth, $R_E$ is the radius of the earth.

The acceleration due to gravity $g$ is independent of the mass $m$ of the body because $m$ doesn’t appear in this equation. We already knew that, but we can now see how it follows from the law of gravitation.

We can measure all the quantities in Eq. (6.4) except for $m_E$, so this relation-ship allows us to compute the mass of the earth. Solving Eq. (6.4) for $m_E$ and using $R_E = 6370$ km $= 6.37 \times 10^6$ m and $g = 9.80$ m/s$^2$, we find

$$m_E = gR_E/G = 5.96 \times 10^{24} \text{ kg}.$$

This is very close to the currently accepted value of $5.912 \times 10^{24} \text{ kg}$. Once Cavendish had measured $G$, he computed the mass of the earth in just this way.

At a point above the earth’s surface a distance $r$ from the center of the earth (a distance $r - R_E$ above the surface), the weight of a body is given by Eq. (6.3) with $R_E$ replaced by $r$:

$$P = F_g = \frac{Gm_E m}{r^2}. $$

(6.5)

The entire assemblage is held together by the mutual gravitational attraction of all the matter in the galaxy.

### 6.3 Gravitational Potential Energy

But Eq. (6.2), $F_g = \frac{Gm_E m}{r^2}$, shows that the gravitational force exerted by the earth (mass $m_E$) does in general depend on the distance $r$ from the body to
the earth’s center. For problems in which a body can be far from the earth’s surface, we need a more general expression for gravitational potential energy.

We consider a body of mass \( m \) outside the earth, and first compute the work \( A_{\text{grav}} \) done by the gravitational force when the body moves directly away from or toward the center of the earth

\[
A_{\text{grav}} = \int_{r_1}^{r_2} F_r \, dr,
\]

where \( F_r \) is the radial component of the gravitational force \( F \) - that is, the component in the direction outward from the center of the earth. Because \( F \) points directly inward toward the center of the earth, \( F_r \) is negative. It differs from Eq. (13.2), the magnitude of the gravitational force, by a minus sign:

\[
F_r = -\frac{Gm_e m}{r^2}.
\]

Substituting Eq. (6.7) into Eq. (6.6), we see that \( E_{\text{grav}} \) is given by

\[
E_{\text{grav}} = -Gm_e m \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{Gm_e m}{r_2} - \frac{Gm_e m}{r_1}.
\]

We now define the corresponding potential energy \( E \) so that \( A_{\text{grav}} = E_1 - E_2 \). Comparing this with Eq. (6.8), we see that the appropriate definition for gravitational potential energy is

\[
E = -\frac{Gm_e m}{r}.
\]
6.4 Kepler’s Laws and the Motion of Planets

The name *planet* comes from a Greek word meaning “wanderer,” and indeed the planets continuously change their positions in the sky relative to the background of stars. One of the great intellectual accomplishments of the 16th and 17th centuries was the threefold realization that the earth is also a planet, that all planets orbit the sun, and that the apparent motions of the planets as seen from the earth can be used to determine their orbits precisely.

The first and second of these ideas were published by Nicolaus Copernicus in Poland in 1543. The nature of planetary orbits was deduced between 1601 and 1619 by the German astronomer and mathematician Johannes Kepler, using precise data on apparent planetary motions compiled by his mentor, the Danish astronomer Tychy Brahe. By trial and error, Kepler discovered three empirical laws that accurately described the motions of the planets:

1. **Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.**
2. A line from the sun to a given planet sweeps out equal areas in equal times.
3. The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.

Kepler did not know why the planets moved in this way. Three generations later, when Newton turned his attention to the motion of the planets, he discovered that each of Kepler’s laws can be *derived*; they are consequences of Newton’s laws of motion and the law of gravitation. Let’s see how each of Kepler’s laws arises.

**Kepler’s First Law**

First consider the elliptical orbits described in the Kepler’s first law. Figure 6.4 shows the geometry of an ellipse. The longest dimension is the *major axis*, with half-length *a*; this half-length is called the *semi-major axis*. The sum of the distances from *S* to *P* and from *S* to *P* is the same for all points on the
curve. \( S \) and \( S' \) are the foci (plural of focus). The sun is at \( S \) (not at the center of the ellipse) and the planet is at \( P \); we think of both as points because the size of each is very small in comparison to the distance between them. There is nothing at the other focus, \( S' \).

The distance of each focus from the center of the ellipse is \( ea \), where \( e \) is a dimensionless number between 0 and 1 called the eccentricity. If \( e = 0 \), the two foci coincide and the ellipse is a circle. The actual orbits of the planets are fairly circular; their eccentricities range from 0.007 for Venus to 0.206 for Mercury. (The earth’s orbit has \( e = 0.017 \).) The point in the planet’s orbit closest to the sun is the perihelion, and the point most distant is the aphelion.

Newton showed that for a body acted on by an attractive force proportional to \( r^2 \), the only possible closed orbits are a circle or an ellipse; he also showed that open orbits must be parabolas or hyperbolas. These results can be derived from Newton’s laws and the law of gravitation, to-gether with a lot more differential equations than we’re ready for.

\[ \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \quad (6.14) \]

Kepler’s Second Law

Figure 6.5 shows the Kepler’s second law. In a small time interval \( dt \), the line from the sun \( S \) to the planet \( P \) turns through an angle \( d\theta \). The area swept out is the colored triangle with height \( r \), base length \( rd\theta \), and area \( dA=1/2r^2d\theta \) in Fig. 6.5b. The rate at which area is swept out, \( dA/dt \), is called the sector velocity:
The essence of the Kepler’s second law is that the sector velocity has the same value at all points in the orbit. When the planet is close to the sun, $r$ is small and $d\theta/dt$ is large; when the planet is far from the sun, $r$ is large and $d\theta/dt$ is small.

To see how the Kepler’s second law follows from Newton’s laws, we express $dA/dt$ in terms of the velocity vector $\vec{\theta}$ of the planet $P$. The component of $\vec{\theta}$ perpendicular to the radial line is $\theta_\perp = \theta \sin \phi$. From Fig. 6.5b the displacement along the direction of $\theta_\perp$ during time $dt$ is $rd\theta$, so we also have $\theta_\perp = rd\theta/dt$. Using this relationship in Eq. (6.14), we find

$$\frac{dA}{dt} = \frac{1}{2}rv\sin \phi.$$  \hfill (6.15)

Figure 6.5 - (a) The planet $(P)$ moves about the sun $(S)$ in an elliptical orbit. (b) at the time $dt$ the line $SP$ sweeps out an area

$$dA = \frac{1}{2}(rd\theta)r = \frac{1}{2}r^2 d\theta.$$  \hfill (6.16)

(c) The planet’s speed varies so that the line $SP$ sweeps out the same area $A$ in a given time $t$ regard-less of the planet’s position in its orbit.

Now $r\theta \sin \phi$ is the magnitude of the vector product $\vec{r} \times \vec{\theta}$, which in turn is $1/m$ times the angular momentum $\vec{L} = \vec{r} \times m\vec{\theta}$ of the planet with respect to the sun. So we have

$$\frac{dA}{dt} = \frac{1}{2m} |\vec{r} \times m\vec{\theta}| = \frac{L}{2m}.$$  \hfill (6.16)

Thus the Kepler’s second law that the sector velocity is constant means that the angular momentum is constant! It is easy to see why the angular momentum
of the planet must be constant, the rate of change of $L$ equals the torque of the gravitational force $\mathbf{F}$ acting on the planet:

$$\frac{d\mathbf{L}}{dt} = \mathbf{\tau} = \mathbf{r} \times \mathbf{F}.$$  

In our situation, $\mathbf{r}$ is the vector from the sun to the planet, and the force $\mathbf{F}$ is directed from the planet to the sun. So these vectors always lie along the same line, and their vector product $\mathbf{r} \times \mathbf{\tau}$ is zero. Hence $\frac{d\mathbf{L}}{dt} = 0$. This conclusion does not depend on the $1/r^2$ behavior of the force; angular momentum is conserved for any force that acts always along the line joining the particle to a fixed point. Such a force is called a central force. (Kepler’s first and third laws are valid for a $1/r^2$ force only.) Conservation of angular momentum also explains why the orbit lies in a plane.

The vector $\mathbf{L} = \mathbf{r} \times m\mathbf{\omega}$ is always perpendicular to the plane of the vectors $\mathbf{r}$ and $\mathbf{\omega}$ since $\mathbf{L}$ is constant in magnitude and direction, $\mathbf{r}$ and $\mathbf{\omega}$ always lie in the same plane, which is just the plane of the planet’s orbit.

**Kepler’s Third Law**

We have already derived the Kepler’s third law for the particular case of circular orbits. Equation (6.12) shows that the period of a satellite or planet in a circular orbit is proportional to the $3/2$ power of the orbit radius. Newton was able to show that this same relationship holds for an elliptical orbit, with the orbit radius $r$ replaced by the semi-major axis $a$:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_s}}.$$  

(6.17)

Since the planet orbits the sun, not the earth, we have replaced the earth’s mass $m_E$ in Eq. (6.12) with the sun’s mass $m_S$. Note that the period does not depend on the eccentricity $e$. An asteroid in an elongated elliptical orbit with
semi-major axis $a$ will have the same orbital period as a planet in a circular orbit of radius $a$. The key difference is that the asteroid moves at different speeds at different points in its elliptical orbit (Fig. 6.5c), while the planet’s speed is constant around its circular orbit (Fig. 6.6).

Figure 6.6 - Except at the poles, the reading for an object being weighed on a scale (the apparent weight) is less than the gravitational force of attraction on the object (the true weight). The reason is that a net force is needed to provide a centripetal acceleration as the object rotates with the earth. For clarity, the illustration greatly exaggerates the angle $\theta$ between the true and apparent weight vectors.

### 6.5 Apparent Weight and the Earth’s Rotation

Because the earth rotates on its axis, it is not precisely an inertial frame of reference. For this reason the apparent weight of a body on earth is not precisely equal to the earth’s gravitational attraction, which we will call the true weight $P_{S0}$ of the body. Figure 6.7 is a cutaway view of the earth, showing three observers. Each one holds a spring scale with a body of mass $m$ hanging from it. Each scale applies a tension force $\vec{F}$ to the body hanging from it, and the reading on each scale is the magnitude $F$ of this force. If the observers are unaware of the earth’s rotation, each one thinks that the scale reading equals the weight of the body because he thinks the body on his spring scale is in equilibrium. So each observer thinks that the tension $\vec{F}$ must be opposed by an equal and opposite force $\vec{F}_0$, which we call the apparent weight. But if the
bodies are rotating with the earth, they are not precisely in equilibrium. Our problem is to find the relationship between the apparent weight $\tilde{P}$ and the true weight $\tilde{P}_0$.

This value is the same for all points on the earth’s surface. If the center of the earth can be taken as the origin of an inertial coordinate system, then the body at the north pole really is in equilibrium in an inertial system, and the reading on that observer’s spring scale is equal to $\tilde{P}_0$. But the body at the equator is moving in a circle of radius $R_E$ with speed, and there must be a net inward force equal to the mass times the centripetal acceleration:

$$P_0 - F = \frac{m\mathcal{g}^2}{R_E}. \quad (6.18)$$

So the magnitude of the apparent weight (equal to the magnitude of $F$) is

$$P = P_0 - \frac{m\mathcal{g}^2}{R_E}. \quad (6.19)$$

If the earth were not rotating, the body when released would have a free-fall acceleration $g_0 = P_0 / m$. Since the earth is rotating, the falling body’s actual acceleration relative to the observer at the equator is $g = P / m$. Dividing Eq. (6.19) by $m$ and using these relationships, we find

$$g = g_0 - \frac{\mathcal{g}^2}{R_E}.$$  

To evaluate $\mathcal{g}^2 / R_E$, we note that in 86,164 s a point on the equator moves a distance equal to the earth’s circumference, $2\pi R_E = 2\pi (6.37 \times 10^6) m$. (The solar day, 86,400 s, is $1/365$ longer than this because in one day the earth also completes $1/365$ of its orbit around the sun.) Thus we find

$$\mathcal{g} = \frac{2\pi (6.37 \times 10^6 m)}{86.164 c} = 465 \text{ } m/c,$$

$$\frac{\mathcal{g}^2}{R_E} = \frac{(465 \text{ } m/c)^2}{6.37 \times 10^6 m} = 0.0339 \text{ } m/c^2.$$
So for a spherically symmetric earth the acceleration due to gravity should be about 0.03 m/s\(^2\) less at the equator than at the poles.

At locations intermediate between the equator and the poles, the true weight \(\ddot{P}_0\) and the centripetal acceleration are not along the same line, and we need to write a vector equation corresponding to Eq. (6.18). From Fig. 6.6 we see that the appropriate equation is

\[
\ddot{P} = \ddot{P}_0 - m\ddot{a}_{rad} = m\dddot{g}_0 - m\dddot{a}_{rod}.
\]  

(6.20)

The difference in the magnitudes of \(g\) and \(g_0\) lies between zero and 0.0339 m s\(^2\). As Fig. 13.26 shows, the direction of the apparent weight differs from the direction toward the center of the earth by a small angle \(b\), which is 0.1 or less.

6.6 Black Holes

In 1916 Albert Einstein presented his general theory of relativity, which included a new concept of the nature of gravitation. In his theory, a massive object actually changes the geometry of the space around it. Other objects sense this altered geometry and respond by being attracted to the first object. The general theory of relativity is beyond our scope in this chapter, but we can look at one of its most startling predictions: the existence of black holes, objects whose gravitational influence is so great that nothing—not even light—can escape them. We can understand the basic idea of a black hole by using Newtonian principles.

Think first about the properties of our own sun. Its mass \(M = 1.99 \times 10^{30}\) kg and radius \(R = 6.96 \times 10^8\) m are much larger than those of any planet, but compared to other stars, our sun is not exceptionally massive.

\[
\rho = \frac{M}{V} = \frac{4}{3} \frac{M}{\pi R^3} = \frac{4}{3} \frac{1.99 \times 10^{30}\text{ kg}}{\pi (6.96 \times 10^8\text{ m})^3} = 1410\text{ kg/m}^3.
\]

The sun’s temperatures range from 5800 K at the surface up to 1.5 \times 10^7\text{K} (about 2.7 \times 10^7\text{F}) in the interior, so it surely contains no solids or liquids. Yet
gravitational attraction pulls the sun’s gas atoms together until the sun is, on average, 41% denser than water and about 1200 times as dense as the air we breathe. Now think about the escape speed for a body at the surface of the sun. In Example 13.5 (Section 13.3) we found that the escape speed from the surface of a spherical mass $M$ with radius $R$ is, $\varrho = \sqrt{\frac{2GM}{R}}$. Substituting $M = \rho V = \rho \left( \frac{4}{3} \pi R^3 \right)$ into the expression for escape speed gives

$$\varrho = \sqrt{\frac{2GM}{R^3}} = \sqrt{\frac{8\pi G\rho}{3}} R.$$ (6.21)

Using either form of this equation, you can show that the escape speed for a body at the surface of our sun is $v = 6.18 \times 10^5$ m/s (about 2.2 million km/h, or 1.4 million mi/h). This value, roughly 1500 the speed of light in vacuum, is independent of the mass of the escaping body; it depends on only the mass and radius (or average density and radius) of the sun. Now consider various stars with the same average density $\rho$ and different radii $R$.

Equation (6.21) shows that for a given value of density $\rho$, the escape speed $v$ is directly proportional to $R$. In 1783 the Rev. John Mitchell, an amateur astronomer, noted that if a body with the same average density as the sun had about 500 times the radius of the sun, its escape speed would be greater than the speed of light in vacuum, $c$. With his statement that “all light emitted from such a body would be made to return toward it,” Mitchell became the first person to suggest the existence of what we now call a black hole.

The first expression for escape speed in Eq. (6.21) suggests that a body of mass $M$ will act as a black hole if its radius $R$ is less than or equal to a certain critical radius. How can we determine this critical radius? You might think that you can find the answer by simply setting $\varrho = c$ in Eq. (6.21). As a matter of fact, this does give the correct result, but only because of two compensating errors. The kinetic energy of light is not $mc^2$, and the gravitational potential energy near a black hole is not given by Eq. (6.9). In 1916, Karl Schwarzschild used
Einstein’s general theory of relativity to derive an expression for the critical radius $R_S$, now called the Schwarzschild radius. The result turns out to be the same as though we had set $\varrho = c$ in Eq. (6.29), so

$$c = \sqrt{\frac{2GM}{R_s}}.$$  

Solving for the Schwarzschild radius $R_s$, we find

$$R_s = \sqrt{\frac{2GM}{c^2}}. \quad (6.22)$$

If a spherical, nonrotating body with mass $M$ has a radius less than $R_S$, then nothing (not even light) can escape from the surface of the body, and the body is a black hole (Fig. 6.7). In this case, any other body within a distance $R_S$ of the center of the black hole is trapped by the gravitational attraction of the black hole and cannot escape from it. The surface of the sphere with radius $R_S$ surrounding a black hole is called the event horizon: Since light can’t escape from within that sphere, we can’t see events occurring inside. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other bodies), its electric charge (from the electric forces it exerts on other charged bodies), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information about the body is irretrievably lost when it collapses inside its event horizon.

Figure 6.7 - The surface of the sphere of radius $R_S$ is called the event horizon of the black hole
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Newton’s Laws